Solving the EOM of Schwarzchild metric

Roland Stevenson*

January 27, 2004

This is work done in answering question 1 of Matt Choptuik's PHY387 course at UT Austin.

We are given the Schwarzchild metric:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{1}$$

We are told to calculate the equations for geodesic motion of a massive test-particle in teh equatiorial plane. The equatorial plane allows us to assume we are in the plane defined by $\theta = \frac{\pi}{2}$. This simplifies the equations a bit, since we know $d\theta = 0$. I think you can also say something about a killing vector along the theta direction, but I don't know too much about this yet.

So to solve for the geodesic equations, you calculate the Euler-Lagrange equations:

$$\frac{\partial}{\partial s}(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0 \tag{2}$$

for each of the variables (t, r, θ, ϕ) and get the following 3 equations (no θ since $d\theta = 0$):

$$-(1 - \frac{2M}{r})\frac{dt}{ds} = K1 \tag{3}$$

$$r^2 \frac{d\phi}{ds} = K2 \tag{4}$$

And then this behemoth:

$$\frac{d^2r}{ds^2} = \frac{\alpha}{r(1-\alpha)}(\frac{1}{2}(\frac{dr}{ds})^2 - \frac{K1^2}{2}) + (1-\alpha)\frac{K2^2}{r^3} \tag{5}$$

where $\alpha = \frac{2M}{r}$

K2 it turns out is just the angular momentum, L. This can be seen in Eq. 6.

K1 squared is the Energy squared. As is noted in Wald's Eq. 6.3.14, the E (our K1) can be calculated in terms of r, r', M, and L:

$$\frac{1}{2}E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}(1 - \frac{2M}{r})(\frac{L^2}{r^2} + 1) \tag{6}$$

Note: kappa in Wald equals 1 since we are dealing with timelike geodesics (the ones ordinary matter travels on). Null geodesics are for light.