The Orbiting Dumbbell

Consider two test masses, m_1, m_2 connected by a massless rigid rod of length d, in orbit about a mass, $M >> m_1, m_2$. Let $(x_i, y_i), i = 1, 2$ be the coordinates of the two test masses, let (x_c, y_c) be the coordinates of the dumbbell's center of mass, and let θ be the angle the rod makes with the x-axis. Defining

$$\mu \equiv \frac{m_2}{m_1 + m_2}$$

the distances of the masses from the center of mass are

$$d_1 = \mu\, d \qquad d_2 = (1 - \mu)d$$

then

$$x_i = x_c \pm d_i \cos\theta \qquad y_i = y_c \pm d_i \sin\theta$$

The moment of inertia of the dumbbell about (x_c, y_c) is

$$I = m_1 {d_1}^2 + m_2 {d_2}^2 = \frac{m_1 {m_2}^2}{(m_1 + m_2)^2} {d_1}^2 + \frac{m_2 {m_1}^2}{(m_1 + m_2)^2} {d_2}^2 = \frac{m_1 m_2}{m_1 + m_2} d^2$$

The equations of motion for the body are

$$(m_1 + m_2)\, \mathbf{a_c} = (m_1 + m_2)\, \ddot{\mathbf{r}}_c = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$I\alpha = I\ddot{\theta} = \sum \tau = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2$$

where

$$\mathbf{F}_1 = -\frac{GMm}{{r_1}^3}\,[x_1, y_1]$$

$$\mathbf{F}_2 = -\frac{GMm}{{r_2}^3}\,[x_2, y_2]$$

are the gravitational forces acting on m_1 and m_2 respectively. The translational equations yield:

$$(m_1 + m_2)\ \ddot{x}_c = -GM\left(\frac{m_1}{{r_1}^3} x_1 + \frac{m_2}{{r_2}^3} y_2\right)$$

$$(m_1 + m_2)\ \ddot{y}_c = -GM\left(\frac{m_1}{{r_1}^3} y_1 + \frac{m_2}{{r_2}^3} y_2\right)$$

or

$$\ddot{x}_c = -GM\left(\frac{1 - \mu}{{r_1}^3} x_1 + \frac{\mu}{{r_2}^3} x_2\right)$$

$$\ddot{y}_c = -GM\left(\frac{1 - \mu}{{r_1}^3} y_1 + \frac{\mu}{{r_2}^3} y_2\right)$$

The rotational equation gives:

$$\begin{aligned}
I\ddot{\theta} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_2 \times \mathbf{F}_2 &= -GM\frac{m_1}{{r_1}^3} d_1 \left(\cos\theta y_1 - \sin\theta x_2\right) + GM\frac{m_2 1}{{r_2}^3} d_2 \left(\cos\theta y_2 - \sin\theta x_2\right) \\
&= -GM\frac{m_1}{{r_1}^3} d_1 \left(\cos\theta y_c - \sin\theta x_c\right) + GM\frac{m_2}{{r_2}^3} d_2 \left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\left(\frac{m_2}{{r_2}^3} d_2 - \frac{m_1}{{r_1}^3} d_1\right)\left(\cos\theta y_c - \sin\theta x_c\right) \\
&= GM\frac{m_1 m_2}{m_1 + m_2} d\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)\left(\sin\theta x_c - \cos\theta y_c\right)
\end{aligned}$$

so

$$\ddot{\theta} = \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)(\sin\theta x_c - \cos\theta y_c)$$

Summarizing, we have:

$$\begin{aligned}
\ddot{x}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}x_1 + \frac{\mu}{{r_2}^3}x_2\right) \\
\ddot{y}_c &= -GM\left(\frac{1-\mu}{{r_1}^3}y_1 + \frac{\mu}{{r_2}^3}y_2\right) \\
\ddot{\theta} &= \frac{GM}{d}\left(\frac{1}{{r_1}^3} - \frac{1}{{r_2}^3}\right)(\sin\theta y_c - \cos\theta x_c)
\end{aligned}$$

where

$$\begin{aligned}
\mu &\equiv \frac{m_2}{m_1+m_2} \\
d_1 &= \mu\, d \\
d_2 &= (1-\mu)d \\
x_i &= x_c \pm d_i \cos\theta \\
y_i &= y_c \pm d_i \sin\theta \\
{r_i}^3 &= \left({x_i}^2 + {y_i}^2\right)^{3/2}
\end{aligned}$$

The total (conserved) energy of the system is

$$E_{\text{tot}} = T_{\text{trans}} + T_{\text{rot}} + V_{\text{grav}}$$

where

$$\begin{aligned}
T_{\text{trans}} &\equiv \frac{1}{2}(m_1+m_2)\left(\dot{x}_c^2 + \dot{y}_c^2\right) \\
T_{\text{rot}} &\equiv \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\, d^2\, \dot{\theta}^2 \\
V_{\text{grav}} &\equiv -GM\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)
\end{aligned}$$