MOTIVATION:

WALD: "DEEP" MOTIVATION; WANT TO QUANTIZE GR QUANTIZATION PROCEDURES - I.E. CONSTRUCTION of QUANTUM FIELD THEORY - GENERALLY START FROM CLASSICAL FIELD THEORY IN EITHER LAGRANGIAN OR HAMILTONIAN FORM

LAGRANGIAN: PATH INTEGRAL QUANTIZATION

HAMILTONIAN: CANONICAL QUANTIZATION

US: ELEGANT / CONCISE / MNEMONIC WAY TO BUILD MODELS; DERIVE EQUATIONS of MOTION (EOM) INVOLVING FUNDAMENTAL FIELDS (TENSOR FIELDS ON A MANIFOLD)

(WHAT'S A FUNDAMENTAL FIELD? SOMETHING YOU CAN WRITE DOWN A LAGRANGIAN FOR)

LAGRANGIAN FORMULATION of CLASSICAL FIELD THEORIES

M: MANIFOLD (4-d OF MOST (ASTRO-)PHYSICAL INTEREST)

ψ: COLLECTION of FIELDS ON M (SUPPRESS ALL INDICES: THOSE ENUMERATING FIELDS THEMSELVES, THOSE ENUMERATING SPECIFIC COMPONENTS OF SPECIFIC FIELDS)

$S[\psi]$: FUNCTIONAL of ψ

$$\text{FIELD CONFIGS ON } M \xrightarrow{S[\psi]} \mathbb{R}$$

ψ_λ : SMOOTH 1-PAR. FAMILY of FIELD CONFIGURATIONS, "STARTS" FROM $\psi_0 = \psi_{\lambda=0}$; EACH ψ_λ SATISFIES APPPROPRIATE BOUNDARY CONDITIONS

(VARIATION) $\delta\psi$: $$\delta\psi \equiv \left.\frac{d\psi}{d\lambda}\right|_{\lambda=0}$$

FUNCTIONAL DERIVATIVES

NOW CONSIDER ALL 1-PAR. FAMS ψ_λ WHICH START FROM ψ_0. REQUIRE

1) $\left.\frac{dS}{d\lambda}\right|_{\lambda=0}$ TO EXIST FOR ALL FAMILIES

2) EXISTENCE of TENSOR FIELD, χ, WHICH IS DUAL TO ψ (I.E. IF ψ IS TYPE (k,ℓ), χ IS TYPE (ℓ, k)) AND WHICH SATISFIES

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \int_M \underbrace{\chi\ \delta\psi}_{} \qquad (F.1.1)$$

IMPLIED CONTRACTION OVER ALL TENSOR INDICES

THEN $\chi \equiv$ FUNCTIONAL DERIV. of S AND WE SAY ~~THAT~~ S IS FUNCTIONALLY DIFFERENTIABLE AT ψ_0

NOTATION: $$\mathcal{X} = \frac{\delta S}{\delta \psi}\bigg|_{\psi_0}$$

LAGRANGIAN DENSITY AND THE ACTION

ACTION: FUNCTIONAL S OF THE FORM

$$S[\psi] = \int_M \mathcal{L}(\psi)$$

WHERE $\mathcal{L}$ (THE LAGRANGIAN DENSITY) IS A LOCAL FCN of ψ AND A FINITE # OF DERIVS of ψ:

$$\mathcal{L}|_x = \mathcal{L}\left(\psi(x), \nabla\psi(x), \ldots, \nabla^k\psi(x)\right)$$

(OFTEN $k=1$) SUCH THAT

1) S IS FUNCTIONALLY DIFFERENTIABLE

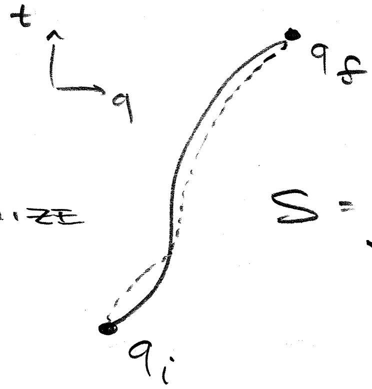
2) FIELD CONFIGURATIONS WHICH EXTREMIZE S, I.E. SO THAT

$$\frac{\delta S}{\delta \psi}\bigg|_{\psi} = 0$$

$\equiv$ FIELD CONFIGURATIONS WHICH SATISFY FIELD EQUATIONS FOR ψ!

• CLEARLY, LAG. FORM of FIELD THEORY CONCEPTUALLY VERY SIMILAR TO LAG. FORM of PARTICLE MECHANICS

PARTICLE MECHANICS :



$$S = \int L(q, \dot{q}, \ldots)\, dt$$

- TO GET E.O.M., EXTREMIZE S SUBJECT TO ADD^L CONSTRAINTS ON PATHS GENERATED BY VARIATION

a) PATHS MUST HAVE FINITE LENGTH (FINITE ACTION)
b) PATHS MUST HAVE FIXED ENDPOINTS q_i, q_f

FIELD THEORY :

- TO GET E.O.M., EXTREMIZE $S[\psi]$ SUBJECT TO

a) $S[\psi] = \int_M \mathcal{L}[\psi] \Rightarrow S[\psi] = \int_U \mathcal{L}[\psi]$

U : COMPACT ("FINITE") REGION of M

b) ONE-PARAMETER FAMILIES, ψ_λ, KEEP VALUE of ψ ON $\underbrace{\partial U \ (\dot{U})}_{\text{BOUNDARY of } U}$ FIXED

EXAMPLE : SCALAR FIELD, ϕ, (KLEIN-GORDON FIELD) IN FLAT SPACETIME

$$\mathcal{L}_{KG} = -\frac{1}{2}\left(\partial_a \phi\, \partial^a \phi + m^2 \phi^2\right)$$

NOTE : OVERALL FACTOR of $\frac{1}{2}$ NOT CRUCIAL "NORMAL-IZATION", BUT "$-$" AND RELATIVE "$+$" BETWEEN TWO TERMS ARE IMPORTANT

MNEMONIC: $L = T - V$ ↘ POTENTIAL

↘ KINETIC ENERGY $\sim +\dot{q}^2$

CONSIDER TIME DERIVS IN $-\frac{1}{2}\partial_a\phi\,\partial^a\phi$

$$= -\frac{1}{2}\eta^{ab}\partial_a\phi\,\partial_b\phi = -\frac{1}{2}(-1)(\partial_t\phi)^2 = +(\partial_t\phi)^2$$

↳ IN GCS = diag(-1,1,1,1)

COMPUTE VARIATION of ACTION

$$\text{"}\delta S_{KG}\text{"} \equiv \left.\frac{dS_{KG}}{d\lambda}\right|_{\lambda=0}$$

$$= \delta\left(\int_U -\frac{1}{2}\left(\partial_a\phi\,\partial^a\phi + m^2\phi^2\right)\right)$$

↳ TREAT AS REGULAR "DERIVATIVE OPERATOR" – I.E. OBEYS LINEARITY, LEIBNITZ, CHAIN RULE etc.

$$= -\int_U \frac{1}{2}\left(\delta(\partial_a\phi)\,\partial^a\phi_0 + \partial_a\phi_0\,\delta(\partial^a\phi_0)\right) + m^2\phi_0\,\delta\phi$$

$$= -\int_U \partial^a\phi_0\,\partial_a(\delta\phi) + m^2\phi_0\,\delta\phi$$

EXERCISE: FILL IN DETAILS, I.E. SHOW THAT

(i) $\delta \equiv \left.\frac{d}{d\lambda}\right|_{\lambda=0}$ COMMUTES WITH ∂_a

(ii) $$\delta(\partial^a\phi\,\partial_a\phi) = 2\partial^a\phi_0\,\delta(\partial_a\phi) = 2\delta(\partial^a\phi)\,\partial_a\phi_0$$

NOW, INTEGRATE BY ~~PARTS~~

$$= \int_{U} (\partial_a \partial^a \phi_0 - m^2 \phi_0)\, \delta\phi \;-\; \int_{\partial U} \partial^a \phi_0\, \delta\phi$$

$= 0$ SINCE ~~RESTRICT~~ ATTN TO VARIATIONS WITH "FIXED B.C's", I.E. ~~SO THAT~~ $\delta\phi|_{\partial U} = 0$

THUS, THE FUNCTIONAL DERIV. OF S_{KG} EXISTS AND IS GIVEN BY

$$\frac{\delta S_{KG}}{\delta \phi} = \partial_a \partial^a \phi - m^2 \phi \equiv \Box\phi - m^2\phi$$

THUS, THE ACTION IS EXTREMIZED $\Leftrightarrow$ E.O.M. SATISFIED WHEN

$$\frac{\delta S_{KG}}{\delta \phi} = 0 \quad \rightarrow \quad \boxed{\Box\phi = m^2\phi}$$

KLEIN-GORDON EQUATION

SIMILARLY, FOR MAXWELL THEORY

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{ab} F^{ab} = -\partial_{[a} A_{b]} \partial^{[a} A^{b]}$$

IS A LAGRANGIAN DENSITY FOR MAX EQNS IN FLAT SPACETIME (EXERCISE: SHOW THIS)

TENSOR DENSITIES

- COMPLICATION IN LAG. FORM OF GR, SINCE IN GR METRIC g_{ab} IS THE FIELD VBL AND THE NATURAL

VOLUME ELEMENT INVOLVED IN $\int_M \cdots$ ABOVE IS

THE 4-FORM ϵ_{abcd} ($\underset{\sim}{\epsilon}$) WHICH SATISFIES

$$\epsilon^{abcd}\,\epsilon_{abcd} = -4!$$

INDICES RAISED WITH g^{ab}; I.E. VOLUME ELEMENT INVOLVES FIELD VBL, MUST BE TAKEN INTO ACCOUNT WHEN EVALUATING FUNCTIONAL DERIVATIVES

STRATEGY: CAN ALWAYS INTRODUCE (AT LEAST LOCALLY) COORD SYSTEM AND ASSOC. COORD. VOLUME ELEMENT e_{abcd} ("$dx\,dy\,dz\,dt$"); IN THE COORD. BASIS WE HAVE

$$e_{\mu\nu\alpha\beta} = \pm 1 \quad \mu\nu\alpha\beta \text{ EVEN/ODD PERM of } 0123$$

$$= 0 \quad \text{OTHERWISE}$$

AND

$$\epsilon_{\mu\nu\alpha\beta} = \sqrt{-g}\, e_{\mu\nu\alpha\beta}$$

$$g = \det[g_{\mu\nu}]$$

THEN, GIVEN THIS VOL. ELEMENT, e_{abcd}, ON M, A TENSOR DENSITY $T^{a\cdots b}{}_{c\cdots d}$ IS A TENSOR WHICH CAN BE WRITTEN

$$T^{a\cdots b}{}_{c\cdots d} = \sqrt{-g}\,\tilde{T}^{a\cdots b}{}_{c\cdots d}$$

WHERE $\tilde{T}^{a\cdots b}{}_{c\cdots d}$ IS A TENSOR WHICH DOES NOT DEPEND ON e_{abcd}

BOTTOM LINE: IN ORDER ~~THAT~~ S_G (ACTION FOR GR) NOT DEPEND ON e_{abcd}, $\mathcal{L}_G$ MUST BE A SCALAR DENSITY; SIM. IN ORDER FOR $dS_G/d\lambda$ TO BE IND. OF e_{abcd}, FUNCTIONAL DERIVS OF S MUST ALSO BE TENSOR DENSITIES

CLAIM: UPTO BOUNDARY TERMS, SCALAR DENSITY

$$\boxed{\mathcal{L}_G = \sqrt{-g}\, R}$$

RICCI SCALAR

IS A LAGRANGIAN DENSITY FOR THE VACUUM EINSTEIN EQN

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2} g_{ab} R = 0$$

(RECALL: $R = g^{ab} R_{ab}$ RICCI TENSOR

$R_{ab} = R_{acb}{}^{c}$ RIEMANN (CURVATURE) TENSOR)

$$S[g^{ab}] = \int \mathcal{L}_G\, \underset{\sim}{e} = \int \mathcal{L}_G\, d^4x = \int \sqrt{-g}\, R\, d^4x$$

DIFF-FORM NOTATION

IS CALLED THE HILBERT ACTION

NOTE: CONVENIENT TO ADOPT INVERSE METRIC g^{ab} AS FUND. FIELD

- CONSIDER 1-PAR VARIATIONS (OF g^{ab}) AS BEFORE (VARS NOW START FROM g^{ab}, I.E. DROP "o" NOTATION)

$$\delta g^{ab} \equiv \left.\frac{dg^{ab}}{d\lambda}\right|_{\lambda=0}$$

ALSO: $g^{ac} g_{cb} = \delta^a{}_b$

$$\rightarrow (\delta g^{ac}) g_{cb} + g^{ac}(\delta g_{cb}) = 0$$

$$\rightarrow \boxed{\delta g_{ab} = -g_{ac} g_{bd} \, \delta g^{cd}}$$

$\delta(\sqrt{-g})$

- IN VARIATION OF $S[g^{ab}]$, WILL NEED $\delta(\sqrt{-g})$; $g^{ab} \delta R_{ab}$
- RECALL (387M, HW 4 KEY), FOR ANY MATRIX M

$$\det(M) = \exp \operatorname{tr} \ln M$$

$$\rightarrow \delta \det(M) = \det(M) \operatorname{tr} M^{-1} \delta M$$

$$\rightarrow \delta(g) = \delta g = g\, g^{ba} \delta g_{ab} = g\, g^{ab} \delta g_{ab}$$

$$\rightarrow \delta(\sqrt{-g}) = -\tfrac{1}{2}(-g)^{-\frac{1}{2}} \delta g$$

$$= \tfrac{1}{2}(-g)(-g)^{-\frac{1}{2}} g^{ab} \delta g_{ab}$$

$$\rightarrow \boxed{\delta(\sqrt{-g}) = \tfrac{1}{2}\sqrt{-g}\, g^{ab} \delta g_{ab} = -\tfrac{1}{2}\sqrt{-g}\, g_{ab} \delta g^{ab}}$$

$g^{ab}\,\delta R_{ab}$

- RECALL: $R_{abc}{}^{d}\omega_d = (\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$ (3.2.3)

$$R_{abc}{}^{d} = -2\partial_{[a}\Gamma^{d}{}_{b]c} + 2\Gamma^{e}{}_{c[a}\Gamma^{d}{}_{b]e} \quad (3.4.3)$$

- ARE NOW CONSIDERING 1-PAR FAMILIES OF METRICS $(g^{ab})_\lambda$ DENOTE $g^{ab}(\lambda)$; CONSIDER ASSOCIATED 1-PARAMETER FAMILY OF COV. DERIVATIVE OPS ${}^{\lambda}\nabla_a$, EACH COMPATIBLE WITH CORR METRIC

$$ {}^{\lambda}\nabla_a g^{bc}(\lambda) = 0$$

$$\nabla_a \equiv {}^{0}\nabla_a \quad (\nabla_a g^{bc}(0) = 0)$$

- DIFFERENCE BETWEEN ${}^{\lambda}\nabla_a$ AND $\nabla_a \cdot {}^{0}\nabla_a$ DEFINES / IS DETERMINED BY TENSOR FIELD $C^{c}{}_{ab}(\lambda)$

$$C^{c}{}_{ab}(\lambda) = \tfrac{1}{2} g^{cd}(\lambda)\{\nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda)\}$$

(7.5.7)

NOTE: BY DEF^{N} $C^{c}{}_{ab}(0) = 0$.

- NOW, FROM

$$R_{abc}{}^{d}(\lambda)\omega_d = ({}^{\lambda}\nabla_a{}^{\lambda}\nabla_b - {}^{\lambda}\nabla_b{}^{\lambda}\nabla_a)\omega_c$$

AND

$$ {}^{\lambda}\nabla_b\omega_c = \nabla_b\omega_c - C^{d}{}_{bc}(\lambda)\omega_d$$

WE CAN EASILY DERIVE (a la DERIVATION of (3.4.3))

$$R_{abc}{}^{d}(\lambda) = {}^{0}R_{abc}{}^{d} - 2\nabla_{[a}C^{d}{}_{b]c} + 2C^{e}{}_{c[a}C^{d}{}_{b]e}$$

(WE'VE SUPRESSED THE λ LABELS ON $C^{a}{}_{bc}$)

• CONTRACTING a:

$$R_{ac}(\lambda) = \underset{(1)}{{}^{\circ}R_{ac}} - \underset{(2)}{2\nabla_{[a} C^{b}{}_{b]c}} + \underset{(3)}{2C^{e}{}_{c[a} C^{b}{}_{b]e}}$$

NOW CONSIDER $\delta R_{ac} = \left.\dfrac{dR_{ac}(\lambda)}{d\lambda}\right|_{\lambda=0}$; CLEARLY

THERE WILL BE NO CONTRIBUTION FROM (1), ALSO, (SCHEMATICALLY), (3) WILL YIELD $\delta C \underbrace{C(0)}_{} = 0$, SO NO CONTRIBUTION FROM (2) EITHER, THUS

$$\delta R_{ac}(\lambda) = -2\nabla_{[a}\delta C^{b}{}_{b]c}$$

WHERE $\delta C^{c}{}_{ab} = \left.\dfrac{dC^{c}{}_{ab}}{d\lambda}\right|_{\lambda=0}$

• FROM (7.5.7) WE HAVE

$$\delta C^{c}{}_{ab} = \tfrac{1}{2} g^{cd}(\nabla_a \delta g_{bd} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ab})$$

$$\to \delta C^{b}{}_{ac} = \tfrac{1}{2} g^{bd}(\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac})$$

$$\to \delta C^{b}{}_{bc} = \tfrac{1}{2} g^{bd}(\nabla_b \delta g_{cd} + \nabla_c \delta g_{bd} - \nabla_d \delta g_{bc})$$

SO $\delta R_{ac} = -\nabla_a \delta C^{b}{}_{bc} + \nabla_b \delta C^{b}{}_{ac}$

$$= \tfrac{1}{2} g^{bd}\big(-\cancel{\nabla_a\nabla_b \delta g_{cd}} - \nabla_a\nabla_c \delta g_{bd} + \cancel{\nabla_a\nabla_d \delta g_{bc}}$$

$$+ \nabla_b\nabla_a \delta g_{cd} + \nabla_b\nabla_c \delta g_{ad} - \nabla_b\nabla_d \delta g_{ac}\big)$$

$$= g^{bd}\left(-\tfrac{1}{2}\nabla_a\nabla_c\,\delta g_{bd} - \tfrac{1}{2}\nabla_b\nabla_d\,\delta g_{ac} + \nabla_b\nabla_{(a}\,\delta g_{c)d}\right)$$

- RELABELING

$$\delta R_{ab} = g^{cd}\left(-\tfrac{1}{2}\nabla_a\nabla_b\,\delta g_{cd} - \tfrac{1}{2}\nabla_c\nabla_d\,\delta g_{ab} + \nabla_c\nabla_{(a}\,\delta g_{b)d}\right)$$

CONTRACTING (RECALL $\nabla_a g_{bc} = \nabla_a g^{bc} = 0$)

$$g^{ab}\delta R_{ab} = -\tfrac{1}{2}g^{cd}\nabla^b\nabla_b\,\delta g_{cd} - \tfrac{1}{2}g^{ab}\nabla^c\nabla_c\,\delta g_{ab} + \nabla^d\nabla^b\,\delta g_{bd}$$

$$= \nabla^a\nabla^b\,\delta g_{ab} - \nabla^a g^{cd}\nabla_a\,\delta g_{cd}$$

$$\boxed{g^{ab}\,\delta R_{ab} = \nabla^a v_a}$$

WHERE $\boxed{v_a \equiv \nabla^b\delta g_{ab} - g^{cd}\nabla_a\,\delta g_{cd}}$

- WE ARE NOW SET UP TO COMPUTE THE VARIATION OF THE HILBERT ACTION

NOTATION	WALD	$\int \cdots \epsilon$	$\int \cdots e$
	US	$\int \cdots dV$	$\int \cdots d^4x$
		$\int \cdots \sqrt{-g}\,d^4x$	

$$S_G = \int \mathcal{L}_G\, d^4x = \int \sqrt{-g}\,R\, d^4x = \int \sqrt{-g}\,g^{ab}R_{ab}\, d^4x$$

$$\delta\mathcal{L}_G = \left.\frac{d\mathcal{L}_G}{d\lambda}\right|_{\lambda=0} = R\,\delta(\sqrt{-g}) + \sqrt{-g}\,R_{ab}\,\delta g^{ab} + \sqrt{-g}\,g^{ab}\,\delta R_{ab}$$

$$\delta(\sqrt{-g}) = -\tfrac{1}{2}\sqrt{-g}\,g_{ab}\,\delta g^{ab}$$

$$\delta S_G = \left.\frac{dS_G}{d\lambda}\right|_{\lambda=0} = \underbrace{\int \nabla^a v_a\,\sqrt{-g}\,d^4x}_{(1)} + \underbrace{\int \left(R_{ab} - \tfrac{1}{2}g_{ab}R\right)\delta g^{ab}\sqrt{-g}\,d^4x}_{(2)}$$

- FIRST TERM IS INTEGRAL OF A DIVERGENCE W.R.T NATURAL VOLUME ELEMENT $dV = \sqrt{-g}\, d^4x$, CAN CONVERT TO BOUNDARY (∂U) INTEGRAL VIA STOKES THM.

NOTE: RESULTING BOUNDARY TERM DOES NOT VANISH FOR GENERAL VARIATIONS WHERE g^{ab} IS HELD FIXED ON ∂U; NEED TO FIX BOTH g^{ab} AND $\partial_c g^{ab}$ FOR TERM TO VANISH

SEE WALD FOR DETAILED DISCUSSION OF BOUNDARY TERM ΔS_G WHICH MUST BE ADDED TO S_G TO "CANCEL" BOUND. TERM ABOVE

- HERE, WE WILL IGNORE BDRY TERM (EQUIVALENTLY, DEMAND THAT g_{ab} AND $\partial_c g_{ab}$ BE FIXED ON ∂U)

- EXTREMIZATION of ACTION $\Rightarrow \delta S_G = 0$

$$\Rightarrow \boxed{R_{ab} - \tfrac{1}{2} g_{ab} R = G_{ab} = 0} \quad \text{VACUUM EINSTEIN EQN}$$

- ALSO SEE WALD FOR DISCUSSION OF PALATINI ACTION

$$\mathcal{L}_G = \mathcal{L}_G[g^{ab}, C^a{}_{bc}] = \mathcal{L}_G[g^{ab}, \nabla_c]$$

YIELDS BOTH EINSTEIN EQN AND METRIC COMPATIBILITY CONDITION $\nabla_c g_{ab} = 0$ $(C^a{}_{bc} = 0)$?

COUPLING TO MATTER (NON-VACUUM CASE)

PRESCRIPTION: CONSTRUCT TOTAL LAGRANGIAN DENSITY $\mathcal{L}$ VIA

$$\mathcal{L} = \mathcal{L}_G + \alpha_M \mathcal{L}_M$$

α_M – CONSTANT (ARBITRARY – CHOSEN "BY CONVENTION")

$\mathcal{L}_M$ – MATTER LAG. DENSITY (SUITABLE FOR CURVED S.T)

$\mathcal{L}_G$ – GRAV. LAG. DENSITY $= \sqrt{-g}\, R$

(MINIMAL) PRESCRIPTION FOR CONSTRUCTING CURVED S.T. $\mathcal{L}_M$ (MINIMAL COUPLING)

- START FROM MINKOWSKI (FLAT S.T.) LAG. DENSITY ${}^{\circ}\mathcal{L}_M$
  1) $\eta_{ab} \to g_{ab}$; $\eta^{ab} \to g^{ab}$
  2) $\partial_a \to \nabla_a$
  3) MULTIPLY BY $\sqrt{-g}$ (MAKE INTO PROPER DENSITY)

EXAMPLES:

1) $$\mathcal{L}_{KG} = -\frac{1}{2}\sqrt{-g}\left(g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\right)$$

2) $$\mathcal{L}_{EM} = -\frac{1}{4}\sqrt{-g}\, g^{ac} g^{bd} F_{ab} F_{cd}$$

$$= -\sqrt{-g}\, g^{ac} g^{bd} \nabla_{[a}A_{b]} \nabla_{[c}A_{d]}$$

- MATTER FIELD EQUATIONS OF MOTION: DERIVED FROM EXTREMIZATION OF ACTION W.R.T VARIATION of MATTER FIELDS (FUNCTIONAL DERIVS. W.R.T. MATTER FIELDS)

EXAMPLE: KLEIN-GORDON FIELD

RECALL: $$\nabla_a T^a = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\, T^a\right) \qquad (3.4.10)$$

$$\to \Box\phi = \nabla_a\nabla^a\phi = \nabla^a\nabla_a\phi = \frac{1}{\sqrt{-g}}\partial_b\left(\sqrt{-g}\, g^{ab}\partial_a\phi\right)$$

$$S_{KG} = -\frac{1}{2}\int \sqrt{-g}\left(g^{ab}\partial_a\phi\,\partial_b\phi + m^2\phi\right) d^4x$$

- VARY W.R.T ϕ

$$\delta S_{KG} = -\int \sqrt{-g}\left(g^{ab}\partial_a\phi\,\partial_b(\delta\phi) + m^2\phi\,\delta\phi\right) d^4x$$

- INTEGRATION BY PARTS ($\delta\phi = 0$ ON $\partial\mathcal{M}$)

$$= \int \left(\partial_b\left(\sqrt{-g}\,g^{ab}\partial_a\phi\right) - \sqrt{-g}\,m^2\phi\right)\delta\phi\, d^4x$$

$$= 0 \quad \text{(EXTREMIZATION of ACTION)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{-g}}\partial_b\left(\sqrt{-g}\,g^{ab}\partial_a\phi\right) = m^2\phi \quad \Rightarrow \quad \boxed{\Box\phi = m^2\phi}$$

"COVARIANT KG EQN"

- EINSTEIN EQUATION (EOM FOR g_{ab}): DERIVED FROM EXTREMIZATION OF TOTAL ACTION W.R.T VARIATION δg_{ab}

FIRST RECALL THAT IF $S = \int \mathcal{L}\, d^4x$, THEN

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \int \frac{\delta S}{\delta g^{ab}}\,\delta g^{ab}\, d^4x \qquad \text{DEFINES}$$

THE FUNCTIONAL DERIVATIVE $\dfrac{\delta S}{\delta g^{ab}}$

- NOW CONSIDER

$$S = S_{TOTAL} = \int \mathcal{L}\, d^4x = \int \left(\mathcal{L}_G + \alpha_M \mathcal{L}_M\right) d^4x$$

• IGNORING BOUNDARY TERMS AS BEFORE, WE HAVE

$$\delta S = \int \left\{ \left(R_{ab} - \tfrac{1}{2} g_{ab} R \right) \delta g^{ab} \sqrt{-g} + \alpha_M \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} \right\} d^4x$$

$$= \int \left\{ R_{ab} - \tfrac{1}{2} g_{ab} R - 8\pi \left(-\frac{\alpha_M}{8\pi} \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \right) \right\} \sqrt{-g}\, \delta g^{ab}\, d^4x$$

NOW, MAKING THE IDENTIFICATION

$$\boxed{T_{ab} = -\frac{\alpha_M}{8\pi} \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}} \qquad (E.1.26)$$

WHICH MAY BE VIEWED AS THE DEFINITION OF THE MATTER-FIELD STRESS TENSOR, WE HAVE THE GENERAL EINSTEIN EQN

$$\boxed{G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R = 8\pi T_{ab}}$$

• ALTERNATE FORM of (E.1.26) (MTW) - LET L_M DENOTE THE MATTER-FIELD LAGRANGIAN SCALAR, I.E. $L_M = \mathcal{L}_M / \sqrt{-g}$

EXAMPLE: $L_{KG} = -\tfrac{1}{2} \left(\nabla^a \phi \nabla_a \phi + m^2 \phi^2 \right)$

THEN $S_M = \int L_M \sqrt{-g}\, d^4x$; VARY W.R.T g^{ab}

$$\delta S_M = \int \left\{ \frac{\partial L_M}{\partial g^{ab}} \delta g^{ab} \sqrt{-g} - \tfrac{1}{2} \sqrt{-g}\, g_{ab}\, \delta g^{ab} L_M \right\} d^4x$$

$$= \int \left\{ \sqrt{-g} \frac{\partial L_M}{\partial g^{ab}} - \tfrac{1}{2} \sqrt{-g}\, g_{ab} L_M \right\} \delta g^{ab}\, d^4x$$

$$\Rightarrow \quad \frac{\delta S_M}{\delta g^{ab}} = \sqrt{-g}\,\frac{\partial L_M}{\partial g^{ab}} - \frac{1}{2}\sqrt{-g}\,g_{ab} L_M$$

so
$$\boxed{T_{ab} = \frac{\alpha_M}{8\pi}\left(-\frac{\partial L_M}{\partial g^{ab}} + \frac{1}{2} g_{ab} L_M \right)} \qquad (E.1.26')$$

EXAMPLE: K.G.; TAKE $\alpha_M = 16\pi$

$$\rightarrow T_{ab} = -2\,\frac{\partial L_{KG}}{\partial g^{ab}} + g_{ab} L_{KG}$$

$$= -2\left(-\frac{1}{2}\nabla_a \phi \nabla_b \phi \right) - \frac{1}{2} g_{ab}\left(\nabla^c \phi \nabla_c \phi + m^2 \phi^2 \right)$$

$$= \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab}\left(\nabla^c \phi \nabla_c \phi + m^2 \phi^2 \right)$$

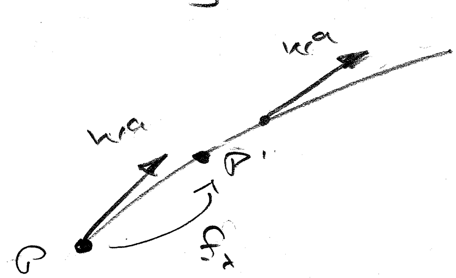
(COMPARE WITH 4.3.10)

CONSERVATION of T_{ab} AS A CONSEQUENCE of GENERAL COVARIANCE (COORD. INDEPENDENCE) AND THE MATTER FIELD EQUATION

RECALL: (APP. C): ONE PARAMETER FAMILY of DIFFEOMORPHISMS (COORD. TRANSFORMATIONS): f_λ

$f_\lambda : M \rightarrow M$ (PREVIOUSLY $\phi_t : M \rightarrow M$)

GENERATED BY VECTOR FIELD w^a



- ACTION of f_λ : "SLIDES MANIFOLD ALONG ITSELF" VIA INTEGRAL CURVES (ORBITS) of w^a ; ALLOWS US TO IDENTIFY / COMPARE DISTINCT ~~EVENTS / TANGENT SPACES~~) TENSOR FIELDS (SUCH AS P, P' IN DIAGRAM)

- IN LIMIT AS $\lambda \to 0$, NATURALLY GET NOTION of LIE DERIVATIVE

$$\mathcal{L}_w T^{\cdots}{}_{\cdots} = \lim_{\lambda \to 0} \left\{ \frac{f^*_{-\lambda} T^{\cdots}{}_{\cdots} - T^{\cdots}{}_{\cdots}}{\lambda} \right\}$$

RECALL: f^*_λ : "PULLBACK of f_λ" MAPS TENSORS FROM ONE TANGENT SPACE TO ANOTHER $\{ (f^* v)(g) = v(g \circ f)$
$v \in V(p)$, $f^* v \in V_{f(p)} \}$

→ ACTION of INFINITESIMAL COORD TRANS (DIFFEO)
$\equiv$ LIE DERIVATIVE ALONG GENERATING VECTOR FIELD

- CONSIDER ACTION of SUCH A FAMILY ON METRIC

$$\delta g^{ab} = \left. \frac{dg^{ab}(\lambda)}{d\lambda} \right|_{\lambda = 0} = \mathcal{L}_w g^{ab} = 2 \nabla^{(a} w^{b)}$$

- NOW CONSIDER THE MATTER ACTION

$$S_M = \int \mathcal{L}_M [g^{ab}, \underbrace{\psi}_{\text{MATTER FIELDS}}] \, d^4x$$

AND DEMAND ~~THAT~~ IT BE INVARIANT UNDER CUR DIFFEOS

$$S_M [g^{ab}, \psi] = S_M [f^*_\lambda g^{ab}, f^*_\lambda \psi]$$

THEN

$$\left.\frac{dS_M}{d\lambda}\right|_{\lambda=0} = 0 = \int \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} d^4x + \int \frac{\delta S_M}{\delta \psi} \delta \psi \, d^4x$$

(1) (2)

- SUPPOSE THAT ψ SATISFIES THE MATTER FIELD EQN^S

THEN

$$\left.\frac{\delta S_M}{\delta \psi}\right|_{\psi} = 0$$

AND TERM (2) DROPS OUT

- ALSO, UPTO A CONSTANT, WE HAVE (E.1.26)

$$\frac{\delta S^M}{\delta g^{ab}} \propto \sqrt{-g}\, T_{ab}$$

SO WE GET

$$0 = \int_U \sqrt{-g}\, T_{ab} \nabla^{(a} w^{b)} d^4x$$

$$= \int_U T_{ab} \nabla^a w^b \, dV$$

NOW ASSUME THAT w^b HAS "COMPACT SUPPORT"; I.E. RESTRICT DIFFEOS TO BE NON-TRIVIAL (I.E NOT THE IDENTITY) IN A FINITE REGION WITHIN U, THEN INT. BY PARTS GIVES

$$0 = -\int_U (\nabla^a T_{ab}) w^b dV + \int_{\partial U} \cancel{T_{ab} w^b} \, dS$$

$$\Rightarrow \qquad \nabla^a T_{ab} = 0$$

- SO, GIVEN COORD.-INDEPENDENT MATTER ACTION (AUTOMATICALLY TRUE IF $\mathcal{L}_G$ IS SCALAR DENSITY), AND MATTER FIELD-EQUATION, THE STRESS TENSOR DEFINED BY (E.1.126) IS AUTOMATICALLY CONSERVED

- CAN APPLY THIS ARGUMENT TO S_G; GET

$$\nabla^a G_{ab} = 0$$

(EXERCISE), SO CAN VIEW CONTRACTED BIANCHI IDENTITY (= "CONSERVATION OF EINSTEIN") AS RESULT OF COORD. INVARIANCE OF HILBERT ACTION

CORRECTION TO LAST DAY'S DISCUSSION (ANSWER TO EXERCISE)

$$C^c{}_{ab}(\lambda) = \frac{1}{2} g^{cd}(\lambda)\left\{\nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda)\right\}$$

$$\delta C^c{}_{ab}(\lambda) = \left.\frac{d C^c{}_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0}$$

TYPICAL TERM

$$\delta\left(g^{cd}(\lambda)\, \nabla_a g_{bd}(\lambda)\right)$$

$$= \delta g^{cd} \left.\left(\nabla_a g_{bd}(\lambda)\right)\right|_{\lambda=0} = 0$$

$$g^{cd}(\lambda)\, \nabla_a \delta g_{bd} \Big|_{\lambda=0}$$

$$= g^{cd}\, \nabla_a \delta g_{bd}$$