THE 3+1 EINSTEIN EQUATIONS

These notes rework the calculation of the 3+1 equations as presented in *Kinematics and Dynamics of General Relativity*, by J. W. York, Jr., which itself is contained in the volume *Sources of Gravitational Radiation*, edited by L. Smarr. Many calculational details omitted from that source are included here.

1) Foliations and Normals

As before, we consider a spacetime M with metric g_{ab} which is sliced into a foliation $\{\Sigma\}$ defined by the isosurfaces of a scalar field τ (the time parameter). Then the spacelike hypersurfaces are, at least locally, described by a *closed* one-form (dual vector field), Ω_a:

$$\Omega_a = \nabla_a \tau \,. \tag{1}$$

Note that since Ω_a is the gradient of a scalar function, and ∇_a is torsion-free, we have

$$\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]}\tau = 0 \,. \tag{2}$$

The norm of Ω_a is given by

$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2} \,, \tag{3}$$

where α is the lapse function, as previously. Thus we can construct the unit-norm dual-vector field, n_a, via

$$n_a = -\alpha\Omega_a = -\alpha\nabla_a\tau \,, \tag{4}$$

where the sign is chosen so that the associated unit-norm, hypersurface-orthogonal vector field, n^a,

$$n^a = g^{ab} n_b \,, \tag{5}$$

is future-directed. Note that we can view n^a as the 4-velocity field of a congruence of observers moving orthogonally to the slices (not necessarily coordinate-stationary). Such observers will have a 4-acceleration, a^b given by

$$a^b = n^a \nabla_a n^b \,. \tag{6}$$

2) The Projection Tensor and the Spatial Metric

In the derivation of the 3+1 form of the Einstein equations, we will necessarily be interested in decomposing various spacetime tensors into hypersurface-tangential ("spatial") and hypersurface-orthogonal ("temporal") pieces. Determining the "temporal" part of a tensor is straightforward, we simply contract with n^a. York throws in a slight twist by introducing a relative minus depending on whether a vector or covector index is being projected. Thus, for a vector field, W^a, we define

$$W^{\hat{n}} = -W^a n_a \,, \tag{7}$$

and, in general, any upstairs $\hat{n}$ index denotes that the original tensor index has been contracted with $-n_a$. On the other hand, for a dual vector field, W_a, we define

$$W_{\hat{n}} = +W_a n^a \,, \tag{8}$$

and then any downstairs $\hat{n}$ index denotes contraction with $+n^a$. To determine the spatial parts of tensors, it is convenient to introduce the notion of a *projection tensor* which, as the name suggests, projects tensors onto the hypersurface. The mixed form of this two-rank tensor is denoted $\perp^a{}_b$ and is defined by

$$\perp^a{}_b \equiv \delta^a{}_b + n^a n_b \, . \tag{9}$$

Note the relative "+" between the identity tensor and $n^a n_b$ which follows from the Lorenztian signature on spacetime. By construction, we have

$$\perp n^a \equiv \perp^a{}_b \, n^b = (\delta^a{}_b + n^a n_b)\, n^b = n^a - n^a = 0 \, , \tag{10}$$

where we have also introduced the notation that a $\perp$ with no indices, operating on an arbitrary tensor expression, means apply the projection tensor to very free tensor index in the expression. Thus, for example

$$\perp S^a{}_{bc} \equiv \perp^a{}_d \perp^e{}_b \perp^f{}_c \, S^d{}_{ef} \, . \tag{11}$$

Any tensor which has had all its free indices projected in this manner is called a *spatial tensor.* It is worth emphasizing the rather obvious point that $\perp$ applied to any tensor expression of the form "tensor product of n^a (or n_a) and something else", vanishes. We will use this fact many times in the following.

If we apply the projection tensor to the spacetime metric g_{ab} itself (which is clearly the same thing as lowering an index on the projection tensor) we get the (spatial) metric, γ_{ab}, on the hypersurfaces:

$$\gamma_{ab} = g_{ab} + n_a n_b \, . \tag{12}$$

Similarly, the contravariant form of the spatial metric is given by

$$\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b \, . \tag{13}$$

Note that *all* tensor indices continue to be raised and lowered with the *spacetime* metric, g_{ab}, and that γ_{ab} and γ^{ab} are *not* inverses. (In fact, of course, the mixed form, $\gamma^a{}_b$, of the spatial metric is just the projection tensor $\perp^a{}_b$.) We also have

$$\mathrm{Tr} \perp \equiv \perp^a{}_a = \delta^a{}_a + n^a n_a = 4 - 1 = 3 \, . \tag{14}$$

Note also, however, that *spatial* tensors can equally well have their indices raised and lowered with γ_{ab}.

3) The Spatial Derivative Operator and Curvature Tensor

We can also use the projection tensor to define a natural derivative operator, D_a, for spatial tensors. Formally, we define

$$D_a \equiv \perp \nabla_a \, , \tag{15}$$

so that for a scalar field ψ, for example, we have

$$D_a \psi \equiv \perp \nabla_a \psi = \perp^b{}_a \nabla_b \psi \, , \tag{16}$$

while for a (spatial) vector field, W^a

$$D_a W^b \equiv \perp \nabla_a W^b = \perp^c{}_a \perp^b{}_d \nabla_c W^d \, . \tag{17}$$

The action of D_a on an arbitrary spatial tensor is then defined in the obvious fashion. D_a is the natural derivative operator for spatial tensors since it is compatible with the spatial metric, i.e.

$$D_a \gamma_{bc} = \perp \nabla_a \gamma_{bc} = \perp \nabla_a (g_{bc} + n_b n_c) = \perp \nabla_a (n_b n_c) = \perp (n_c \nabla_a n_b + n_b \nabla_a n_c) = 0 \,. \tag{18}$$

$D_a \gamma^{bc} = 0$ follows from an exactly parallel computation, or, more directly, simply by raising indices (using either metric!) on the above expression.

The intrinsic curvature of the three-dimensional hypersurfaces is given by the Riemann tensor associated with the spatial metric and is denoted $\mathcal{R}_{abc}{}^d$. It may be defined via its action on an arbitrary spatial dual-vector, W_a:

$$(D_a D_b - D_b D_a) W_c = \mathcal{R}_{abc}{}^d W_d \,. \tag{19}$$

$\mathcal{R}_{abc}{}^d$ is, of course, a spatial-tensor itself, and hence satisfies

$$\mathcal{R}_{abc}{}^d n^a = \mathcal{R}_{abc}{}^d n^b = \mathcal{R}_{abc}{}^d n^c = \mathcal{R}_{abc}{}^d n_d = 0 \,. \tag{20}$$

In addition, $\mathcal{R}_{abcd}$ has the usual symmetries:

$$\mathcal{R}_{abcd} = \mathcal{R}_{[ab]cd} = \mathcal{R}_{ab[cd]} \,, \tag{21}$$

$$\mathcal{R}_{[abc]d} = 0 \,, \tag{22}$$

and

$$\mathcal{R}_{abcd} = R_{cdab} \,. \tag{23}$$

Finally, we can contruct the spatial Ricci tensor, $\mathcal{R}_{ab}$, and spatial Ricci scalar, $\mathcal{R}$, in the usual manner

$$\mathcal{R}_{ab} = \mathcal{R}_{acb}{}^c \,, \tag{24}$$

$$\mathcal{R} = \mathcal{R}_a{}^a \,. \tag{25}$$

4) The Extrinsic Curvature Tensor

The embedding of the slices in the spacetime is described by the *extrinsic curvature* tensor. Before defining this tensor and discussing its properties, we establish two useful results concerning derivatives of the normal vector field. The first of these is

$$\perp \nabla_{[a} n_{b]} = 0 \,. \tag{26}$$

To see this, start from

$$\begin{aligned}
\perp \nabla_a n_b &= (\delta^c{}_a + n^c n_a)(\delta^d{}_b + n^d n_b) \nabla_c n_d \\
&= \nabla_a n_b + n_a n^c \nabla_c n_b + n_b n^d \nabla_a n_d + n_a n^c n_b n^d \nabla_c n_d \\
&= \nabla_a n_b + n_a n^c \nabla_c n_b & (27) \\
&= \nabla_a n_b + n_a a_b \,, & (28)
\end{aligned}$$

where we have used

$$n^d \nabla_a n_d = n_d \nabla_a n^d = \frac{1}{2} \nabla_a \left(n^d n_d \right) = \frac{1}{2} \nabla_a (-1) = 0 \,, \tag{29}$$

to eliminate the last two terms in the second line. We now consider each of the two terms of (27) in turn. Using (4), we have

$$\nabla_a n_b = -\nabla_a (\alpha \Omega_b) = -(\nabla_a \alpha)\ \Omega_b - \alpha (\nabla_a \Omega_b) = -(\nabla_a \alpha)\ \Omega_b - \alpha (\nabla_a \nabla_b \tau)\ , \tag{30}$$

so that

$$\nabla_{[a} n_{b]} = -(\nabla_{[a} \alpha)\, \Omega_{b]}\,. \tag{31}$$

Next, again using (4), we have

$$\begin{aligned} n_a n^c \nabla_c n_b &= \alpha^2 \Omega_a \Omega^c \left(-(\nabla_c \alpha)\, \Omega_b - \alpha (\nabla_c \Omega_b)\right) \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_c \Omega_b \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \alpha^3 \Omega_a \Omega^c \nabla_b \Omega_c \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - \frac{\alpha^3}{2} \Omega_a \nabla_b \left(-\alpha^{-2}\right) \\ &= -\alpha^2 \Omega^c (\nabla_c \alpha)\, \Omega_a \Omega_b - (\nabla_b \alpha)\ \Omega_a\,, \end{aligned} \tag{32}$$

where we have used (2) in going from the second to third line, and (3) in going from the third to fourth. Thus,

$$n_{[a} n^c \nabla_c n_{b]} = -(\nabla_{[b} \alpha)\, \Omega_{a]} = +(\nabla_{[a} \alpha)\, \Omega_{b]}\,, \tag{33}$$

which, when combined with (31), immediately establishes (26) from (27).

Our second useful preliminary result relates the 4-acceleration a_b to the derivative of the lapse function. Specifically, we have

$$a_b = D_b \ln \alpha\,. \tag{34}$$

To see this, we reexpress the right and left hand sides to show that they are indeed equal. On the one hand we have from (32), and again using (4),

$$a_b = n^c \nabla_c n_b = \alpha \Omega^c (\nabla_c \alpha)\, \Omega_b + \alpha^{-1} \nabla_b \alpha\,, \tag{35}$$

while on the other we have

$$D_b \ln \alpha = \perp^c{}_b \nabla_c \ln \alpha = (\delta^c{}_b + n^c n_b) \left(\alpha^{-1} \nabla_c \alpha\right) = \alpha \Omega^c (\nabla_c \alpha)\, \Omega_b + \alpha^{-1} \nabla_b \alpha\,. \tag{36}$$

Recalling our first preliminary result (26), we now define the extrinsic curvature tensor, K_{ab}:

$$K_{ab} = K_{(ab)} = -\perp \nabla_{(a} n_{b)} = -\perp \nabla_a n_b\,. \tag{37}$$

Using this definition and (28), we have

$$\nabla_a n_b = -K_{ab} - n_a a_b\,, \tag{38}$$

which rather explicitly displays the decomposition of the derivative of the normal field into a hypersurface-tangential piece—the extrinsic curvature, and a hypersurface-orthogonal piece—the 4-acceleration.

A definition of K_{ab} which is equivalent to (37) can be made in terms of the Lie derivative along the normal vector field. Specifically, we have

$$K_{ab} = -\frac{1}{2} \pounds_n \gamma_{ab} = -\frac{1}{2} \perp \pounds_n g_{ab}\,. \tag{39}$$

To see this, we note that we have from (37) and (38)

$$K_{ab} = K_{(ab)} = -(\nabla_{(a}n_{b)} + n_{(a}a_{b)})\,, \tag{40}$$

while $\pounds_n \gamma_{ab}$ can be written as

$$\begin{aligned}
\pounds_n \gamma_{ab} &= n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a n^c + \gamma_{ac} \nabla_b n^c \\
&= n^c \nabla_c \left(g_{ab} + n_a n_b\right) + \left(g_{cb} + n_c n_b\right) \nabla_a n^c + \left(g_{ac} + n_a n_c\right) \nabla_b n^c \\
&= n^c \nabla_c \left(n_a n_b\right) + \nabla_a n_b + \nabla_b n_a \\
&= 2\left(\nabla_{(a} n_{b)} + n_{(a} a_{b)}\right) = -2K_{ab}\,,
\end{aligned} \tag{41}$$

where we have used (29) (two times) in going from the second line to the third. Also,

$$\pounds_n g_{ab} = n^c \nabla_c g_{ab} + g_{cb} \nabla_a n^c + g_{ac} \nabla_b n^c = \nabla_a n_b + \nabla_b n_a = 2\nabla_{(a} n_{b)}\,, \tag{42}$$

and then

$$K_{ab} = -\frac{1}{2} \perp \pounds_n g_{ab}\,, \tag{43}$$

follows immediately from (37). Finally, we note that since the extrinsic curvature is a spatial tensor, we of course have

$$n^a K_{ab} = 0\,. \tag{44}$$

This last result will also be used often in the sequel.

5) The Gauss-Codazzi Equations

We now begin computing projections of the 4-dimensional Riemann curvature tensor, R_{abcd}, starting with $\perp R_{abcd}$. To this end, we first consider the 4-dimensional Ricci identity as applied to a spatial dual-vector, v_a:

$$v^a \perp R_{abcd} = \perp (v^a R_{abcd}) = \perp (R^a{}_{bcd} v_a) = \perp (R_{dcb}{}^a v_a) = \perp \left(\nabla_d \nabla_c v_b - \nabla_c \nabla_d v_b\right)\,, \tag{45}$$

where

$$n^a v_a = 0\,. \tag{46}$$

We have

$$\begin{aligned}
\perp \nabla_c v_b &= \nabla_c v_b + n_b n^f \nabla_c v_f + n_c n^e \nabla_e v_b + n_c n^e n_b n^f \nabla_e v_f \\
&= \nabla_c v_b - n_b v_f \nabla_c n^f + n_c n^e \nabla_e v_b - n_c n_b v_f a^f\,,
\end{aligned} \tag{47}$$

where we have used (6) and

$$n^f \nabla_e v_f = -v_f \nabla_e n^f\,, \tag{48}$$

which follows from applying ∇_b to (46). Continuing, we have

$$\begin{aligned}
\perp (\nabla_d \perp \nabla_c v_b) = D_d D_c v_b &= \perp \nabla_d \nabla_c v_b + \perp \nabla_d \left(n^c n^e \nabla_e v_b - n_b v_f \nabla_c n^f - n_c n_b v_f a^f\right) \\
&= \perp \nabla_d \nabla_c v_b - \perp (\nabla_d n_b)(\nabla_c n_f) v^f \\
&= \perp \nabla_d \nabla_c v_b - K_{db} K_{ca} v^a\,,
\end{aligned} \tag{49}$$

or

$$\perp \nabla_d \nabla_c v_b = D_d D_c v_b + K_{db} K_{ca} v^a\,. \tag{50}$$

Thus we have

$$\begin{aligned}
\perp(R^a{}_{bcd}v_a) &= \perp(\nabla_d\nabla_c v_b - \nabla_c\nabla_d v_b)\\
&= D_dD_cv_b - D_cD_dv_b + K_{db}K_{ca}v^a - K_{cb}K_{da}v^a\\
&= (\mathcal{R}_{dcba} + K_{db}K_{ca} - K_{cb}K_{da})\,v^a\\
&= (\mathcal{R}_{abcd} + K_{db}K_{ca} - K_{cb}K_{da})\,v^a\\
&= v^a \perp R_{abcd}\,. \qquad (51)
\end{aligned}$$

or

$$\perp R_{abcd} = \mathcal{R}_{abcd} + K_{db}K_{ca} - K_{cb}K_{da}\,. \qquad (52)$$

We now wish to compute $\perp R_{abc\hat{n}}$; applying the Ricci identity to n^a and projecting, we have

$$\begin{aligned}
\perp R_{\hat{n}bcd} &= \perp(R_{abcd}n^a) = \perp(R_{dcba}n^a)\\
&= \perp(\nabla_d\nabla_c n_b - \nabla_c\nabla_d n_b)\\
&= \perp(\nabla_d(K_{cb} + n_c a_b) - \nabla_c(K_{db} + n_d a_b))\\
&= \perp(\nabla_d K_{cb} - \nabla_c K_{db} + (\nabla_d n_c - \nabla_c n_d)a_b)\\
&= \perp(\nabla_d K_{cb} - \nabla_c K_{db})\\
&= D_dK_{cb} - D_cK_{db}\,. \qquad (53)
\end{aligned}$$

where we have used (38) in going from the second to third line and (26) in going from the fourth to fifth. Relabeling indices, and using the symmetries of Riemann, we have

$$\perp R_{abc\hat{n}} = D_bK_{ac} - D_aK_{bc}\,. \qquad (54)$$

Equations (52) and (54) are known as the *Gauss-Codazzi* equations.

6) The Constraint Equations

We are now nearly ready to derive the constraint equations. We begin by noting that, as is easily verified, a generic type (0,2) symmetric tensor, $\sigma_{ab} = \sigma_{(ab)}$ has the following 3+1 decomposition:

$$\sigma_{ab} = \perp\sigma_{ab} - 2n_{(a}\perp\sigma_{b)\hat{n}} + n_an_b\sigma_{\hat{n}\hat{n}}\,. \qquad (55)$$

We define the following projections of the stress tensor, T_{ab}

$$\rho \equiv T_{\hat{n}\hat{n}} = T_{ab}n^an^b\,, \qquad (56)$$

$$j^a \equiv \perp T^{a\hat{n}} = -\perp(T^{ab}n_b)\,, \qquad (57)$$

$$S_{ab} \equiv \perp T_{ab}\,. \qquad (58)$$

ρ, j^a and S_{ab} may be interpreted as the local energy density, momentum density and spatial stress tensor, respectively, as measured by observers moving orthogonally to the slices.

We now consider

$$\begin{aligned}
\perp R_{ab} &= \perp(g^{cd}R_{acbd})\\
&= \perp(\gamma^{cd}R_{acbd}) - \perp(n^cn^dR_{acbd})\\
&= \perp(\gamma^{cd}R_{acbd}) - \perp R_{a\hat{n}b\hat{n}}\,. \qquad (59)
\end{aligned}$$

Now, clearly, by the same argument that allowed us to write (45):

$$v^a \perp R_{abcd} = \perp (v^a R_{abcd}) \,, \tag{60}$$

where v^a is an arbitrary spatial vector, we have

$$\perp (\gamma^{cd} R_{acbd}) = \gamma^{cd} \perp R_{acbd} = g^{cd} \perp R_{acbd} \,. \tag{61}$$

Thus we find

$$\perp R_{ab} = g^{cd} \perp R_{acbd} - \perp R_{a\hat{n}b\hat{n}} \,. \tag{62}$$

Now using the general 3+1 decomposition formula (55) for a symmetric tensor, we have

$$R_{a\hat{n}b\hat{n}} = \perp R_{a\hat{n}b\hat{n}} - 2n_{(a} \perp \mathcal{R}_{b)\hat{n}\hat{n}\hat{n}} + n_a n_b R_{\hat{n}\hat{n}\hat{n}\hat{n}} \,. \tag{63}$$

Since R_{abcd} is antisymmetric on its first two or last two indices, the last two terms in the above decomposition vanish, and we have

$$\perp R_{a\hat{n}b\hat{n}} = R_{a\hat{n}b\hat{n}} \,. \tag{64}$$

Contracting (62) and using this last result, we find

$$g^{ab} \perp R_{ab} = -R_{\hat{n}\hat{n}} + g^{ab} g^{cd} \perp R_{acbd} \,. \tag{65}$$

We can derive another expression for $g^{ab} \perp R_{ab}$ by starting from the 3+1 decomposition (55) applied to R_{ab} (and slightly rearranged):

$$\perp R_{ab} = R_{ab} + 2n_{(a} \perp R_{b)\hat{n}} - n_a n_b R_{\hat{n}\hat{n}} \,. \tag{66}$$

Contracting, and using the fact that $n^a \perp v_a = 0$ for *any* dual-vector v_a, we find

$$g^{ab} \perp R_{ab} = R + R_{\hat{n}\hat{n}} \,. \tag{67}$$

Equating (65) and (67) and solving for R, we have

$$R \;=\; -2R_{\hat{n}\hat{n}} + g^{ab} g^{cd} \perp R_{acbd} \,. \tag{68}$$

Now consider the Einstein field equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \,, \tag{69}$$

and contract both indices with the normal field, n^a, to produce what could be called the "purely temporal" Einstein equation:

$$G_{ab} n^a n^b = R_{ab} n^a n^b - \frac{1}{2} g_{ab} n^a n^b R = 8\pi T_{ab} n^a n^b \,, \tag{70}$$

or, using (68) and (56)

$$R_{\hat{n}\hat{n}} + \frac{1}{2} R \;=\; R_{\hat{n}\hat{n}} + \frac{1}{2}(-2R_{\hat{n}\hat{n}} + g^{ab} g^{cd} \perp R_{acbd}) \;=\; \frac{1}{2} g^{ab} g^{cd} \perp R_{acbd} \;=\; 8\pi\rho \,. \tag{71}$$

But from the first of the Gauss-Codazzi equations (52) we have

$$g^{ab} g^{cd} \perp R_{acbd} \;=\; g^{ab} g^{cd} (\mathcal{R}_{acbd} + K_{ab} K_{cd} - K_{ad} K_{bc}) \;=\; \mathcal{R} + K^2 - K^a{}_b K^b{}_a \,. \tag{72}$$

where

$$K \equiv K^a{}_a\,, \tag{73}$$

is the trace of the extrinsic curvature tensor. Thus, we find from (71) and (72)

$$\mathcal{R} + K^2 - K^a{}_b K^b{}_a = 16\pi\rho\,, \tag{74}$$

which is known as the *Hamiltonian constraint.*

We now consider the Einstein equation in the form

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = 8\pi T^{ab}\,, \tag{75}$$

and contract one index with $-n_a$ (recall the convention given by equation (7)):

$$G^{a\hat{n}} = R^{a\hat{n}} + \frac{1}{2} n^a R = 8\pi T^{a\hat{n}}\,. \tag{76}$$

Projecting the remaining index onto the hypersurface and using the definition (57) of the momentum density, we have

$$\perp G^{a\hat{n}} = \perp R^{a\hat{n}} = 8\pi \perp T^{a\hat{n}} = 8\pi j^a\,. \tag{77}$$

Following a development precisely analogous to (61)–(62), we find

$$\perp R_{a\hat{n}} = g^{cd} \perp R_{ac\hat{n}d} - \perp R_{a\hat{n}\hat{n}\hat{n}} = -g^{cd} \perp R_{acd\hat{n}}\,. \tag{78}$$

Using the second of the Gauss-Codazzi equations (54), this becomes

$$\perp R_{a\hat{n}} = -g^{cd}\left(D_c K_{ad} - D_a K_{cd}\right) = D_a K - D^b K_{ab}\,. \tag{79}$$

Raising the remaining free index we have (again recalling (7))

$$\perp R^{a\hat{n}} = \perp G^{a\hat{n}} = D_b K^{ab} - D^a K\,. \tag{80}$$

Thus, we find

$$D_b K^{ab} - D^a K = 8\pi j^a\,, \tag{81}$$

which is known as the *momentum constraint.*

The crucial feature of the constraint equations (74) and (81), is that they involve *only* spatial tensors (including spatial derivatives of spatial tensors)—in particular, they do *not* involve *explicit time derivatives* of spatial tensors. Thus, these equations *are* equations of constraint which must be satisfied by the fundamental 3+1 variables, γ_{ab} and K_{ab} at all times (i.e. on all slices).

7) Time and Time Derivatives

In this section we establish some results concerning certain vector fields and Lie derivatives along these vector fields. The idea here is to introduce sufficiently general notions of "time" and "time derivatives while maintaining a geometric approach.

We first prove two results concerning Lie derivatives along the normal vector field, n^a, of *spatial* type $(0,l)$ tensors (spatial covariant tensors). The first result states that if $S_{a_1 a_2 \cdots a_l}$ is a spatial tensor, so that

$$n^{a_i}\, S_{a_1 \cdots a_l} = 0 \qquad i = 1, 2, \cdots, l\,, \tag{82}$$

then $\pounds_n S_{a_1 \cdots a_l}$ is also a spatial tensor. Denoting the general type $(0,l)$ spatial tensor by $\mathbf{S}$ we can thus write

$$\perp \pounds_n \mathbf{S} = \pounds_n \mathbf{S}\,. \tag{83}$$

The proof is straightforward. We have

$$\pounds_n S_{a_1 \cdots a_l} \;=\; n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c)\, S_{a_1 \cdots c \cdots a_l}\,. \tag{84}$$

Now contract the jth index with n^a:

$$n^{a_j}\, \pounds_n S_{a_1 \cdots a_l} \;=\; n^{a_j} n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c)\, n^{a_j} S_{a_1 \cdots c \cdots a_l}\,. \tag{85}$$

Now, because $S_{a_1 \cdots c \cdots a_l}$ is spatial, all of the terms in the sum (the "correction terms") vanish except when $i = j$ (i.e. when the jth index of $S_{a_1 \cdots c \cdots a_l}$ is being corrected. Also, we can use $n^{a_i}\, S_{a_1 \cdots a_l} = 0$ to throw the derivative in the first term onto n^{a_j}. Thus, we have

$$n^{a_j}\, \pounds_n S_{a_1 \cdots a_l} \;=\; -n^c \nabla_c n^{a_j} S_{a_1 \cdots a_l} + n^c \nabla_c n^{a_j} S_{a_1 \cdots a_l} = 0\,, \tag{86}$$

and since this holds for arbitrary $j = 1, \cdots l$, we have established (83). Our second result is that if $S_{a_1 a_2 \cdots a_l}$ is spatial, and f is an arbitrary function, then

$$\pounds_{fn} S_{a_1 a_2 \cdots a_l} = f\, \pounds_n S_{a_1 a_2 \cdots a_l} \tag{87}$$

Again, the proof is straightforward:

$$\begin{aligned}
\pounds_{fn} S_{a_1 a_2 \cdots a_l} \;&=\; f n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} \nabla_{a_i} (f n^c)\, S_{a_1 \cdots c \cdots a_l} \\
&=\; f n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} \left((\nabla_{a_i} f)\, n^c + f\, (\nabla_{a_i} n^c) \right) S_{a_1 \cdots c \cdots a_l} \\
&=\; f \left(n^c \nabla_c S_{a_1 \cdots a_l} + \sum_{i=1}^{l} (\nabla_{a_i} n^c)\, S_{a_1 \cdots c \cdots a_l} \right) \\
&=\; f\, \pounds_n S_{a_1 a_2 \cdots a_l}\,.
\end{aligned} \tag{88}$$

Below we will argue that the vector field, N^a defined by

$$N^a = \alpha\, n^a \tag{89}$$

is a *natural* orthogonal vector field with which to Lie-differentiate tensors in computing general time derivatives. This vector field has the important property that

$$\pounds_N \perp^a{}_b = 0 \tag{90}$$

which implies that if $\mathbf{S}$ is *any* spatial tensor (not necessarily type $(0,l)$), then $\pounds_N \mathbf{S}$ is also spatial:

$$\perp \pounds_N \mathbf{S} = \pounds_N \mathbf{S} \tag{91}$$

To see that (91) follows from (90), note that for a general type (k,l) spatial tensor we have

$$S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l} \;=\; \perp^{a_1}{}_{c_1} \perp^{a_2}{}_{c_2} \cdots \perp^{a_k}{}_{c_k} \perp^{d_1}{}_{b_1} \perp^{d_2}{}_{b_2} \cdots \perp^{d_l}{}_{b_l} S^{c_1 c_2 \cdots c_k}{}_{d_1 d_2 \cdots d_l} \tag{92}$$

Now, applying $\pounds_N$ to both sides of this expression, and using the Liebnitz rule, we easily see that, given (90), the only term which survives is the one where the Lie derivative acts on $S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l}$:

$$\pounds_N S^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l} \;=\; \perp^{a_1}{}_{c_1} \perp^{a_2}{}_{c_2} \cdots \perp^{a_k}{}_{c_k} \perp^{d_1}{}_{b_1} \perp^{d_2}{}_{b_2} \cdots \perp^{d_l}{}_{b_l} \pounds_N S^{c_1 c_2 \cdots c_k}{}_{d_1 d_2 \cdots d_l} \,, \tag{93}$$

and this is precisely (91). It remains to show that (90) is true. We have

$$\begin{aligned}
\pounds_N \perp^a{}_b &= N^c\,\nabla_c \perp^a{}_b \;-\; \perp^c{}_b \nabla_c N^a \;+\; \perp^a{}_c \nabla_b N^c \\
&= (\alpha n^c)\,\nabla_c\left(\delta^a{}_b + n^a n_b\right) - \left(\delta^c{}_b + n^c n_b\right)\nabla_c\left(\alpha n^a\right) + \left(\delta^a{}_c + n^a n_c\right)\nabla_b\left(\alpha n^c\right) \\
&= \alpha n^a n^c \nabla_c n_b + \alpha n_b n^c \nabla_c n^a - \nabla_b\left(\alpha n^a\right) - n_b n^c \nabla_c\left(\alpha n^a\right) + \nabla_b\left(\alpha n^a\right) + n^a n_c \nabla_b\left(\alpha n^c\right) \\
&= \alpha n^a a_b + \alpha n_b a^a - \alpha n_b a^a - n_b n^a n^c \nabla_c \alpha + \alpha n^a n_c \nabla_b n^c - n^a \nabla_b \alpha \\
&= \alpha n^a \left(a_b - \alpha^{-1}\left(\nabla_b \alpha + n_b n^c \nabla_c \alpha\right)\right) \\
&= \alpha n^a \left(a_b - \alpha^{-1} D_b \alpha\right) = \alpha n^a \left(a_b - D_b \ln \alpha\right) = 0\,.
\end{aligned} \tag{94}$$

where we have used (9), (89), (29) and (34).

Now, recall that our foliation is defined by a closed one-form (dual-vector field), Ω_a:

$$\Omega_a = \nabla_a \tau\,. \tag{95}$$

Since $n^a = -\alpha \Omega^a$ and $\Omega^a \Omega_a = -\alpha^{-2}$, we have

$$N^a \Omega_a = 1\,. \tag{96}$$

It is this normalization which makes N^a the *natural orthogonal* vector field to use in computing "time derivatives" (i.e. for use in Lie differentiation). However, there is no justification for restricting attention only to "normal time derivatives" and, in fact, we can and will consider Lie differentiation along other "time directions", t^a, appropriately normalized via

$$t^a \Omega_a = 1\,, \tag{97}$$

by adding to N^a an arbitrary *spatial* vector β^a (which is just the *shift vector* we have previously discussed):

$$t^a = N^a + \beta^a = \alpha n^a + \beta^a\,, \tag{98}$$

$$\beta^a n_a = 0\,. \tag{99}$$

8) The Evolution Equations

In order to derive the 3+1 evolution equations, we have to compute the projection of one more piece of the spacetime curvature tensor, namely $\perp R_{a\hat{n}b\hat{n}}$. Starting from the Ricci identity applied to n_a and using (38) and (6), we have

$$
\begin{aligned}
\perp R_{a\hat{n}b\hat{n}} &= \perp \left(n^c \left(\nabla_b \nabla_c n_a - \nabla_c \nabla_b n_a\right)\right) \\
&= \perp \left(n^c \nabla_c \left(K_{ba} + n_b a_a\right) - n^c \nabla_b \left(K_{ca} + n_c a_a\right)\right) \\
&= \perp \left(n^c \nabla_c K_{ba} + a_b a_a - n^c \nabla_b K_{ca} + \nabla_b a_a\right) \ ,
\end{aligned}
\tag{100}
$$

Now, since $n^c K_{ca} = 0$, we have

$$
-n^c \nabla_b K_{ca} = \nabla_b n^c K_{ca} \, . \tag{101}
$$

Thus, using this result, adding and subtracting $\nabla_a n^c K_{bc}$, and noting that from (37) we have $- \perp \nabla_a n^c K_{bc} = K_a{}^c K_{bc}$, we find

$$
\begin{aligned}
\perp R_{a\hat{n}b\hat{n}} &= \perp \left(n^c \nabla_c K_{ba} + \nabla_b n^c K_{ca} + \nabla_a n^c K_{bc} - \nabla_a n^c K_{bc} + a_b a_a + \nabla_b a_a\right) \\
&= \perp \left(£_n K_{ab} + K_a{}^c K_{bc} + a_a a_b + \nabla_b a_a\right) \ .
\end{aligned}
\tag{102}
$$

Now, using $a_b = D_b \ln \alpha$, we have

$$
\begin{aligned}
\perp \left(a_a a_b + \nabla_b a_a\right) &= \perp \Big(D_a \ln \alpha \ D_b \ln \alpha + \nabla_b \left(\alpha^{-1} \nabla_a \alpha\right)\Big) \\
&= \perp \Big(\alpha^{-2} D_a \alpha \ D_b \alpha - \alpha^{-2} D_b \alpha \ D_a \alpha + \alpha^{-1} \left(\nabla_b D_a \alpha\right)\Big) \\
&= \alpha^{-1} D_b D_a \alpha = \alpha^{-1} D_a D_b \alpha
\end{aligned}
\tag{103}
$$

(the torsion-free property of D_a used in the last step follows directly from the torsion free proerty of ∇_a.) In addition, using the two preliminary results (82) and (87) from the beginning of this section we have

$$
\perp £_n K_{ab} = £_n K_{ab} = \alpha^{-1} £_N K_{ab} \, . \tag{104}
$$

Using (103) and (104), (102) becomes

$$
\perp R_{a\hat{n}b\hat{n}} = \alpha^{-1} £_N K_{ab} + K_{ac} K^c{}_b + \alpha^{-1} D_a D_b \alpha \, . \tag{105}
$$

8a) The Evolution Equations for the Spatial Metric

The evolution equations for the spatial metric are essentially identities which follow from the definition (39) of the extrinsic curvature:

$$
K_{ab} = -\frac{1}{2} £_n g_{ab} \tag{106}
$$

However, as discussed above, for full generality, we wish to use Lie-differentiation along the vector field

$$
t^a = N^a + \beta^a = \alpha n^a + \beta^a \tag{107}
$$

as our "time derivative". Again using (87), as well as a fundamental property of the Lie derivative for arbitrary vector fields v^a and w^a, and arbitrary tensor fields $\mathbf{S}$:

$$
£_{v+w} \mathbf{S} = £_v \mathbf{S} + £_w \mathbf{S} \, , \tag{108}
$$

we have

$$\begin{aligned} \pounds_t \gamma_{ab} &= \pounds_N \gamma_{ab} + \pounds_\beta \gamma_{ab} \\ &= \alpha \pounds_n \gamma_{ab} + \pounds_\beta \gamma_{ab} \,. \end{aligned} \tag{109}$$

or

$$\pounds_t \gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta \gamma_{ab} \,. \tag{110}$$

8b) The Evolution Equations for the Extrinsic Curvature

We first observe that the Einstein equations

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \,, \tag{111}$$

may be contracted to yield

$$G = -R = 8\pi T \,. \tag{112}$$

Thus, the field equations may be rewritten as

$$R_{ab} \;=\; 8\pi T_{ab} + \frac{1}{2} g_{ab} R = 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \,. \tag{113}$$

Projecting onto the hypersurface, we have

$$\perp R_{ab} = 8\pi \left(\perp T_{ab} - \frac{1}{2} \gamma_{ab} T \right) \,. \tag{114}$$

Now from definitions (56)–(58), as well as our expression (55) for the 3+1 decomposition of a general, symmetric, type $(0,2)$ tensor, we find

$$\perp T_{ab} \;\equiv\; S_{ab} \;=\; T_{ab} + 2 n_{(a} \perp T_{b)\hat{n}} - n_a n_b T_{\hat{n}\hat{n}} \,. \tag{115}$$

Contracting, we get

$$S = T + T_{\hat{n}\hat{n}} \,, \tag{116}$$

or, using (56)

$$T = S - \rho \,. \tag{117}$$

Thus, (114) becomes

$$\perp R_{ab} = 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} \left(S - \rho \right) \right) \tag{118}$$

Now, from (62), we have

$$\perp R_{ab} = - \perp R_{a\hat{n}b\hat{n}} + g^{cd} \perp R_{acbd} \,. \tag{119}$$

Using (52) and (105), this becomes

$$\begin{aligned} \perp R_{ab} &= -\left(\alpha^{-1} \pounds_N K_{ab} + K_{ac} K^c{}_b + \alpha^{-1} D_a D_b \alpha \right) + g^{cd} \left(\mathcal{R}_{abcd} + K_{ab} K_{cd} - K_{ad} K_{cb} \right) \\ &= -\alpha^{-1} \pounds_N K_{ab} - 2 K_{ac} K^c{}_b - \alpha^{-1} D_a D_b \alpha + \mathcal{R}_{ab} + K K_{ab} \,. \end{aligned} \tag{120}$$

Equating (118) and (120), and using

$$\pounds_N K_{ab} = \pounds_{t-\beta} K_{ab} = \pounds_t K_{ab} - \pounds_\beta K_{ab} \,, \tag{121}$$

we solve for $\pounds_t K_{ab}$ to get our evolution equations for the extrinsic curvature:

$$\pounds_t K_{ab} = \pounds_\beta K_{ab} - D_a D_b \alpha + \alpha \left(\mathcal{R}_{ab} + K K_{ab} - 2 K_{ac} K^c{}_b - 8\pi \left(S_{ab} - \frac{1}{2}\gamma_{ab} \left(S - \rho \right) \right) \right) . \quad (122)$$

We can derive an alternate version of this evolution equation which involves the "mixed" form, $K^a{}_b$ of the extrinsic curvature, which has been used by several researchers in the past, and which we will tend to use in the course. We start from

$$\alpha^{-1} \pounds_N K_{ab} - 2 K_{ac} K^c{}_b - \alpha^{-1} D_a D_b \alpha + \mathcal{R}_{ab} + K K_{ab} = 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} \left(S - \rho \right) \right) , \quad (123)$$

and note that because all of the tensors appearing in this expression are spatial, we can raise indices with γ^{ab} to get:

$$\alpha^{-1} \gamma^{ac} \pounds_N K_{ab} - 2 K^{ac} K_{cb} - \alpha^{-1} D^a D_b \alpha + R^a{}_b + K K^a{}_b = 8\pi \left(S^a{}_b - \frac{1}{2} \delta^a{}_b + n^a n_b \left(S - \rho \right) \right) . \quad (124)$$

Now

$$\begin{aligned} \pounds_N K^a{}_b &= \pounds_N \left(\gamma^{ac} K_{cb} \right) \\ &= K_{cb} \pounds_N \gamma^{ac} + \gamma^{ac} \pounds_N K_{cb} \\ &= \alpha K_{cb} \pounds_n \gamma^{ac} + \gamma^{ac} \pounds_N K_{cb} \\ &= -2 \alpha K_{cb} K^{ac} + \gamma^{ac} \pounds_N K_{cb} \, , \end{aligned} \quad (125)$$

so

$$\gamma^{ac} \pounds_N K_{cb} = \pounds_N K^a{}_b + 2 \alpha K^{ac} K_{cb} \, . \quad (126)$$

Substituting this result in (124) and using

$$\pounds_N K^a{}_b = \pounds_t K^a{}_b - \pounds_\beta K^a{}_b \, , \quad (127)$$

we find

$$\pounds_t K^a{}_b = \pounds_\beta K^a{}_b - D^a D_b \alpha + \alpha \left(\mathcal{R}^a{}_b + K K^a{}_b + 8\pi \left(\frac{1}{2} \perp^a{}_b \left(S - \rho \right) - S^a{}_b \right) \right) . \quad (128)$$