Instability of an ''approximate black hole''

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We investigate the stability of a family of spherically symmetric static solutions in vacuum Brans-Dicke theory (with $\omega=0$) recently described by van Putten. Using linear perturbation theory, we find one exponentially growing mode for every member of the family of solutions, and thus conclude that the solutions are not stable. Using a previously constructed code for spherically symmetric Brans-Dicke theory, additional evidence for instability is provided by directly evolving the static solutions with perturbations. The full nonlinear evolutions also suggest that the solutions are black-hole-threshold critical solutions. [S0556-2821(97)00810-2]

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Recently van Putten has proposed a one-parameter family (parameter ϵ) of solutions to spherically symmetric Brans-Dicke theory for use in numerical relativity as an approximate black hole [1]. These solutions have the attractive property that for small values of this parameter, the ''exterior'' solution approaches that of Schwarzschild. However, the event horizon of Schwarzschild is replaced with a high redshift horizon and all metric components remain finite at this horizon. In addition to regularizing the horizon, these solutions have a global timelike coordinate.

As van Putten stresses, these solutions could have promise for numerical relativity because of the difficulties that arise when dealing numerically with boundary conditions at the horizon of a black hole. To be useful as approximate black holes, however, the solutions, like Schwarzschild solutions, must be stable.

In this paper, we recompute the solutions considered by van Putten and carry out a linear perturbation analysis about them. In so doing we find, for generic ϵ, modes which grow exponentially in time. We also directly evolve perturbations on the background of these solutions and confirm the instability predicted by linear theory. Thus, although of some theoretical interest, these solutions are unlikely to be of direct use in the context of mocking-up a black hole in general relativity.

To begin, let us review the static family of solutions considered by van Putten. We note that they were first written down by Brans and Dicke [2] but in an isotropic coordinate system as opposed to the Schwarzschild-like coordinates that van Putten uses.

We work in Brans-Dicke theory and assume spherical symmetry. We choose a coordinate system such that the metric has the form

$$ds^2=-e^{\nu(r,t)}dt^2+e^{\lambda(r,t)}dr^2+r^2d\Omega^2. \tag{1}$$

Using $\phi(r,t)$ for the Brans-Dicke field, the field equations are [2]

$$G_{\mu\nu}=\frac{8\pi}{\phi}T_{\mu\nu}, \tag{2}$$

where, in vacuum, the Brans-Dicke stress tensor is given by

$$T_{\mu\nu}=\frac{\omega}{8\pi\phi}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho}\right)+\frac{1}{8\pi}(\phi_{;\mu\nu}-g_{\mu\nu}\Box\phi), \tag{3}$$

and ω is the Brans-Dicke coupling constant [3]. The field ϕ satisfies the covariant wave equation [3]

$$\Box\phi=\frac{8\pi}{2\omega+3}T^{\rm matter}=0. \tag{4}$$

This equation, along with van Putten's restriction to $\omega=0$, simplifies Eq. (3) to $T_{\mu\nu}=\phi_{;\mu\nu}/8\pi$. As the stress tensor is traceless, the field equations may then be written in the simple form

$$R_{\mu\nu}=\frac{\phi_{;\mu\nu}}{\phi}, \tag{5}$$

where $R_{\mu\nu}$ is the usual Ricci curvature tensor.

We introduce a new field $\psi(r,t)$ such that

$$e^{\psi(r,t)}=\frac{A}{r\phi(r,t)}, \tag{6}$$

where A is a constant [4]. The field equations are

$$\begin{aligned}
&\frac{\dot\lambda}{r}+\dot\psi'=\left(\dot\psi+\frac{\dot\lambda}{2}\right)\left(\frac{1}{r}+\psi'\right)+\frac{1}{2}\nu'\dot\psi,\\
&-\frac{e^{\lambda}}{r}=\psi'+\frac{1}{2}(\lambda'-\nu'),\\
&[r\dot\psi e^{(\lambda-\nu)/2-\psi}]^{\cdot}=[(1+r\psi')e^{(\nu-\lambda)/2-\psi}]',\\
&e^{\lambda-\nu}[2\ddot\psi-\dot\psi(2\dot\psi+\dot\nu)]=\left(\frac{1}{r}+\psi'\right)\left(\nu'+\frac{2}{r}\right)\\
&-\frac{1}{r}(\lambda'+\nu'),
\end{aligned} \tag{7}$$

where overdots and primes denote derivatives with respect to t and r, respectively. The first three equations above correspond to the tr and $\theta\theta$ components of Eq. (5), and the wave equation, respectively. The final equation is a convenient linear combination of the tt and rr components of Eq. (5), and the wave equation.

To find the time-independent solutions as van Putten does, all time derivatives appearing in Eqs. (7) are set to zero, yielding

$$\psi' + \frac{1}{2}(\lambda' - \nu') = \frac{-e^{\lambda}}{r},$$

$$\psi' = \frac{1}{r}(2Z-1),$$

$$\lambda' + \nu' = 2Z\left(\frac{2}{r} + \nu'\right), \tag{8}$$

where $Z \equiv 1/2e^{\psi+(\lambda-\nu)/2}$ and is identical to that defined by van Putten [1]. Details regarding the solution of this system of equations can be found in [1]; here we will only quote the results. The metric components are

$$e^{\lambda} = 1 - 4Z + \frac{1}{\epsilon}Z(1-Z) \equiv \frac{1}{\epsilon}(Z_1+Z)(Z_2-Z),$$

$$e^{\nu} = \left[\frac{1-\frac{Z}{Z_2}}{1+\frac{Z}{Z_1}}\right]^{1/(Z_1+Z_2)},$$

where ϵ is an integration constant, and the constants Z_1 and Z_2 are given by

$$Z_1 = 2\epsilon - \tfrac{1}{2} + \frac{1}{2}\sqrt{1-4\epsilon+16\epsilon^2},$$

$$Z_2 = -2\epsilon + \tfrac{1}{2} + \frac{1}{2}\sqrt{1-4\epsilon+16\epsilon^2}. \tag{9}$$

The field Z is found from the transcendental equation

$$\frac{|Z|^{Z_1+Z_2}}{|Z_2-Z|^{Z_1}|Z+Z_1|^{Z_2}} = r^{-(Z_1+Z_2)}. \tag{10}$$

Note that the field ψ can be recovered once Z, λ, and ν are known.

As van Putten points out, Eq. (10) has four solutions, only one of which is Schwarzschild-like in its exterior (van Putten's type Ia [1]). For this solution we have $\epsilon > 0$ and $Z \to Z_2$ as $r \to 0$ while $Z \to 0$ as $r \to \infty$.

It is worthwhile to consider the small r behavior of these fields. In terms of the integration constant ϵ, this behavior is found to be

$$e^{\lambda} \approx \frac{Z_2^2}{\epsilon}\left(\frac{Z_2}{Z_1+Z_2}\right)^{(Z_2/Z_1)-1} r^{Z_2/\epsilon(Z_1+Z_2)} e^{\nu}$$
$$\approx \left(\frac{Z_2}{Z_1+Z_2}\right)^{(Z_2/Z_1)/(Z_1+Z_2)} \left(\frac{Z_1}{Z_1+Z_2}\right)^{1/(Z_1+Z_2)} r^{Z_2/\epsilon}. \tag{11}$$

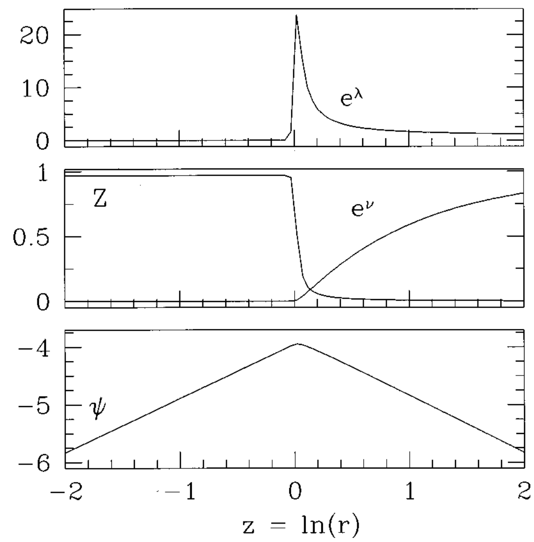

FIG. 1. Unperturbed fields for $\epsilon = 0.01$. As $\epsilon \to 0$, the field Z approaches a step function (see Fig. 3) and the field e^{λ} becomes more sharply peaked. This value of ϵ is chosen to correspond with Fig. 2 of [1].

If in addition to small r, we consider the limit of small ϵ, these expressions reduce to

$$e^{\lambda} \approx \frac{1}{e\epsilon} r^{-5+1/\epsilon}, \quad e^{\nu} \approx \frac{\epsilon}{e} r^{-3+1/\epsilon}. \tag{12}$$

Figure 1 displays the solution to Eqs. (8) subject to the initial conditions derived from Eq. (11) for $\epsilon = 1/100$ (the same value shown in [1]).

Having constructed these static solutions, we can now address the question of their stability. For a given ϵ, such a time-independent solution used in numerical relativity as an approximate black hole, were it not stable, would either collapse to a black hole or possibly disperse leaving flat space.

To investigate stability, we perform a standard linear perturbation analysis. As such, we consider the case that the fields *do* possess a small time-dependent part and make the following expansion for small δ:

$$\psi \to \psi_0(r) + \delta\psi_1(r,t), \quad \nu \to \nu_0(r) + \delta\nu_1(r,t),$$

$$\lambda \to \lambda_0(r) + \delta\lambda_1(r,t), \tag{13}$$

where subscripts 0 and 1 denote the unperturbed and perturbed fields, respectively.

We substitute the expansion (13) into the full set of Eqs. (7), keeping only terms to linear order. Because the unperturbed fields satisfy Eq. (7) by construction, we are left with the linear equations for $(\psi_1, \nu_1, \lambda_1)$:

$$\frac{\dot{\lambda}_1}{r} = \left(\dot{\psi}_1 + \frac{\dot{\lambda}_1}{2}\right)\left(\frac{1}{r} + \psi_0'\right) - \dot{\psi}_1' + \frac{1}{2}\nu_0'\dot{\psi}_1,$$

$$-\frac{e^{\lambda_0}}{r}\lambda_1 = \psi_1' + \frac{1}{2}(\lambda_1' - \nu_1'),$$

$$0=\psi_1''+\left(\frac{1}{r}+\psi_0'\right)\left[\frac{1}{2}(\nu_1'-\lambda_1')-\psi_1'\right]$$
$$+\psi_1'\left[\frac{1}{2}(\nu_0'-\lambda_0')-\psi_0'+\frac{1}{r}\right]-e^{\lambda_0-\nu_0}\ddot{\psi}_1,$$
$$\frac{1}{r}(\lambda_1'+\nu_1')=\left(\frac{1}{r}+\psi_0'\right)\nu_1'+\psi_1'\left(\nu_0'+\frac{2}{r}\right)$$
$$-2e^{\lambda_0-\nu_0}\ddot{\psi}_1. \tag{14}$$

We perform the standard Fourier decomposition of $\psi_1(t,r)$,

$$\hat{\psi}_1(r,\sigma)=\int e^{i\sigma t}\psi_1(r,t)dt, \tag{15}$$

and the other perturbed fields. On substitution into Eq. (14), the defining relation for $\hat{\psi}_1$ then decouples from $\hat{\lambda}_1$ and $\hat{\nu}_1$, yielding a single equation

$$0=\hat{\psi}_1''+\hat{\psi}_1'\frac{1}{r}\left[1+e^{\lambda_0}\frac{3Z_0-1}{Z_0-1}\right]$$
$$+\hat{\psi}_1\left[\sigma^2e^{\lambda_0-\nu_0}-\frac{Z_0}{r(Z_0-1)}\left(\nu_0'+\frac{4Z_0}{r}\right)e^{\lambda_0}\right], \tag{16}$$

which can be solved for the mode $\hat{\psi}_1(r)$. We have thus reduced the perturbation problem to a single one-dimensional ordinary differential equation (ODE) with the undetermined eigenvalue σ^2.

Solution of Eq. (16) requires appropriate boundary conditions on $\hat{\psi}_1(r)$. It is common to enforce regularity at the origin of a spherically symmetric spacetime, but in the current case the unperturbed solution is itself irregular. Thus, assuming regularity of the perturbation might seem improper. However, although ψ_0 is logarithmically divergent at $r=0$, $\exp(\psi_0)$ is regular at the origin, going to zero as a positive power of r. Hence, it is not unreasonable to impose regularity on the field e^{ψ}. Further, because $\exp(\psi)=\exp(\psi_0+\delta\psi_1)=\exp(\psi_0)(1+\delta\psi_1)$, it is reasonable to assume the regularity of $\hat{\psi}_1$ at the origin. At most, the mode $\hat{\psi}_1$ could have a logarithmic divergence, however, if we can find an unstable mode with the stricter criterion of regularity, then the solutions are still, in general, unstable.

Enforcing the assumption of regularity of $\hat{\psi}_1$ allows us to find a series expansion for $\hat{\psi}_1$ near the origin. For very small r, Eq. (16) becomes

$$0=\hat{\psi}_1''+\hat{\psi}_1'\frac{1}{r}+\sigma^2C\hat{\psi}_1r^p. \tag{17}$$

The positive coefficient C is determined from Eq. (11) and depends on ϵ. The exponent $p=(Z_2/\epsilon)(Z_1+Z_2-1)$ likewise depends only on ϵ and in such a way that $p>-2$ for $\epsilon>0$. We can now find an expansion for $r\ll 1$:

$$\hat{\psi}_1(r)=\hat{\psi}_1(0)\left[1-\frac{\sigma^2C}{2+p}r^{2+p}+\left(\frac{\sigma^2C}{2+p}\right)^2r^{4+2p}+\cdots\right].$$

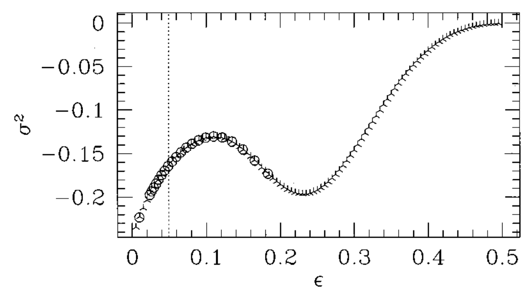

FIG. 2. Plot of the unstable eigenvalues σ^2 versus parameter ϵ. The cross marks and open circles represent data from two independent evaluations of the modes. Their correspondence indicates a quite small uncertainty. The vertical dotted line denotes the value of ϵ for which Figs. 4, 5, and 6 are computed.

Because of the linearity of the problem, $\hat{\psi}_1(0)$ can be arbitrarily chosen as it reflects the scaling in the problem. It turns out that we need to use only the first couple of terms in the expansion to get accurate results.

Given a background solution to Eq. (8) for a particular ϵ, we can now solve the eigenvalue problem (16) for the modes $\hat{\psi}_1(r)$ and corresponding characteristic frequencies σ^2. In our particular case, the instability of the original soliton solutions is indicated by the existence of one or more exponentially growing modes. These are solutions to the perturbation equations with negative eigenvalues: $\sigma^2<0$.

In practice, we integrate the unperturbed equations and the perturbation equation simultaneously from $r\approx 0$ to large r, looking for a solution which has a negative eigenvalue and obeys the boundary conditions. Specifically, we demand that the mode be finite at the origin and vanish asymptotically. We use a standard ODE integrator and standard shooting techniques in our search.

Although our search has not been exhaustive, we generically find precisely one growing mode for each value of ϵ. This is sufficient for us to conclude that the static solutions are unstable. The eigenvalues found for these solutions are shown in Fig. 2.

Having found the perturbation modes, looking at the limit $\epsilon\rightarrow 0$ is instructive. As this limit is approached, the unperturbed solution becomes more and more Schwarzschild-like in the exterior, and this resemblance is precisely the reason why the family has been proposed as a good model of a black hole. With this in mind, one may wonder why Fig. 2 shows that as $\epsilon\rightarrow 0$, there is still a growing mode. Certainly these results do not show Schwarzschild solutions to be unstable; rather we point out that within the ''effective horizon'' of this approximate black hole, the solution is very different from the interior Schwarzschild solution for all ϵ (see Fig. 3). Hence, it is reasonable to assert that the solution is unstable for any ϵ, including the solution $\epsilon=0$.

Further, since it is the case that to an outside observer, the $\epsilon=0$ solution is indistinguishable from that of Schwarzschild, it is logical to assume that any perturbation of the solution will not change the view of this observer. In other words, as $\epsilon\rightarrow 0$, any perturbation should have decreasing support outside the ''effective horizon'' of the approximate black hole. Indeed, we observe this kind of behavior. As one decreases ϵ, the profile of the mode $\hat{\psi}_1$ is seen to approach

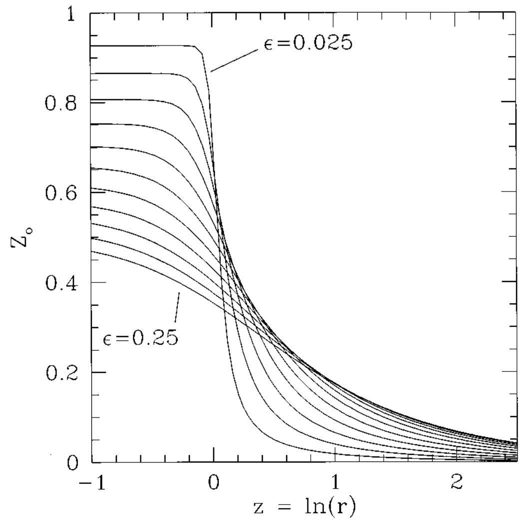

FIG. 3. The unperturbed field $Z_0(r)$ for $0.025 \leqslant \epsilon \leqslant 0.24$ (uniform in ϵ). The field $Z_0(r)$ approaches a step function as ϵ is decreased. The approach of the unperturbed Z_0 to zero at large z signifies the approach to the Schwarzschild solution in that region. For $z<0$, however, the solution clearly is not Schwarzschild-like.

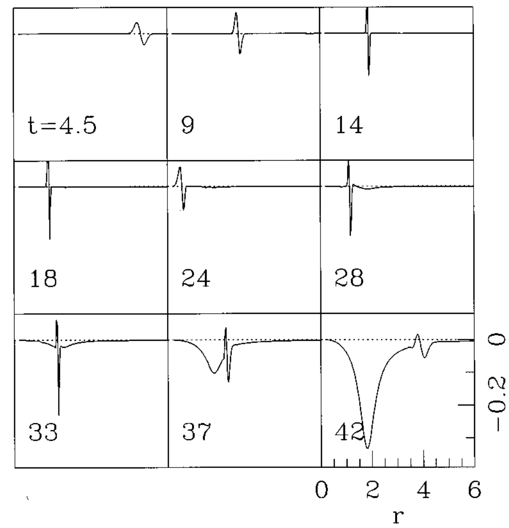

FIG. 4. Series of snapshots of the product $r\Pi_\xi$ (related to the time derivative of ψ) in the Einstein frame. A small perturbation ($\Delta M/M = 0.018\%$) at large radius is introduced to the initially static solution ($\epsilon = 0.127$). The perturbation passes through the singularity at $r=0$ (between $t=24$ and 28) and escapes to $r=\infty$. As the perturbation passes the redshift horizon (as it propagates inwards, the perturbation is seen to experience a blueshift), the excitation of a growing mode is clearly seen.

that of a δ function at the position where the apparent horizon would asymptotically form.

We find additional evidence for instabilitty by evolving the static solutions (with small perturbations added) using the full time-dependent equations of motion [5]. This has the added benefit of providing information concerning the states to which the static solutions evolve when perturbed. To this end, we have adapted a previously developed spherically symmetric code for Brans-Dicke theory [5] which allows us to evolve these solutions. Because the evolutions in [5] are performed in the Einstein conformal frame (as compared to the Brans-Dicke frame in which van Putten works and in which we have worked thus far), we transform the fields used above to the Einstein frame. In this frame, the field equations are equivalent to those for a massless scalar field minimally coupled to gravity.

After recovering the static solutions in the Einstein frame and inputing a transformed solution into the code, we introduce a small ingoing perturbation to the fields at large radius. For generic values of ϵ, we find that van Putten's solution either collapses or disperses after the perturbation reaches the high-redshift horizon. Figure 4 clearly demonstrates the instability for a specific ϵ. In this case, the perturbation induces collapse to a black hole.

In order to facilitate comparison of the perturbation results with those of the fully nonlinear evolution, we first define the quantities below in terms of the fields found in [5]

$$\overline{a_1} = a(r,t) - a(r,0),$$

$$\overline{\Phi_1} = \Phi_\xi(r,t) - \Phi_\xi(r,0),$$

$$\overline{\Pi_1} = \Pi_\xi(r,t) - \Pi_\xi(r,0), \qquad (18)$$

such that fields with a bar and a subscripted 1 denote nonlinear deviations from the unperturbed solution (*nonlinear modes*). By transforming the modes found from linear perturbation theory to the Einstein frame, we may now compare directly the linear modes with the nonlinear modes ($\overline{\Pi_1}$, $\overline{a_1}$, $\overline{\Phi_1}$). In Fig. 5 we show all the modes rescaled to the unit interval. From the near congruence, we conclude that we are observing the actual evolution of the growth of these perturbation modes.

In addition to confirming our perturbative results, the full evolution provides evidence that these static solutions represent critical solutions to black hole formation. By this we mean that these solutions represent a boundary in the space of solutions between those that form black holes and those that do not. To demonstrate this criticality we begin with an

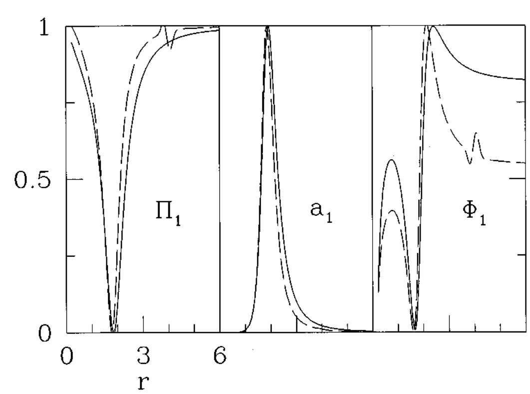

FIG. 5. Comparison of modes computed in perturbation theory (solid lines) with modes computed from the full nonlinear evolution (dashed lines). The nonlinear modes were computed by taking a late-time profile from the evolution and subtracting the initial data for that field, as defined in Eq. (18). Perturbative modes have been numerically transformed to the Einstein frame. All fields are rescaled to the interval $[0 \cdots 1]$ and are plotted with respect to r.

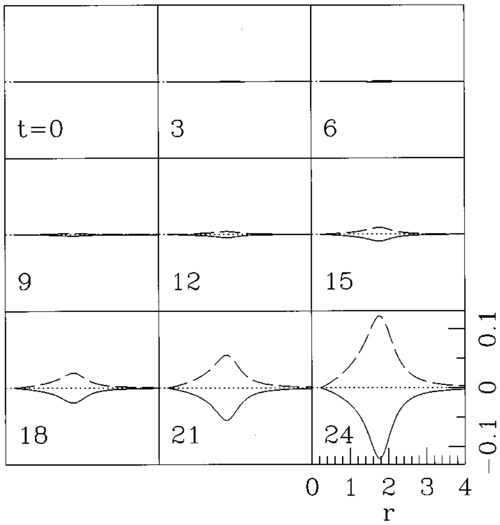

FIG. 6. Uniform-in-time series of the solution when initially perturbed with the predicted mode ($\Delta M/M = 0.004\%$, $\epsilon = 0.127$). Shown here is the field Π_ξ (related to the time derivative of ψ). The evolution shown in solid results in collapse to a black hole. The dashed line shows the evolution resulting from switching the sign of the perturbation at the initial time. That the initial sign of the introduced perturbation separates eventual collapse from dispersal indicates that the unperturbed solution is critical.

unperturbed solution in the Einstein frame and add a perturbation to the fields. In this case instead of the arbitrary perturbation at large r shown in Fig. 4, we add the predicted mode found in the perturbation analysis (solid lines of Fig. 5). When this mode is added with some small positive amplitude, we invariably see collapse of the van Putten approximate black hole to a genuine black hole (see the solid lines in Fig. 6). In contrast, when the perturbation is added with a small negative amplitude, we see dispersal of the solution (see the dashed lines in Fig. 6). Thus it would appear that this solution sits at the threshold between solutions that form black holes and those that disperse [6].

Having found evidence that these are threshold solutions, we are led to ask if they are attracting. For them to represent attracting critical solutions (intermediate attractors) the unstable (relevant) mode which we find must be shown to be unique [7]. Because both the perturbation analysis and the full evolution appear to indicate the presence of only a single unstable mode (see Fig. 5), we suspect that these solutions might well represent an intermediate attractor for black hole formation. We plan to address this issue in future work.

From the results presented here, these solutions would appear to be analagous to the $n=1$ Bartnik-McKinnon (BM) solution in the Einstein-Yang-Mills (EYM) system [8]. This static solution was found to be an intermediate attractor in the gravitational collapse of spherically symmetric SU(2) fields with one side of the threshold being black hole formation and the other dispersion of the Yang-Mills field [9].

After completion of this work, we became aware of other work which had considered solutions similar to those examined here. Static, spherically symmetric solutions to the minimally coupled Einstein-Klein-Gordon equations were studied by Buchdahl [10] and later by Wyman [11]. These solutions were written down in the Einstein frame in contrast to van Putten who works in vacuum Brans-Dicke (which is conformally equivalent to the Einstein-Klein-Gordon system). In addition, Jetzer and Scialom were able to show that Wyman's solutions are generically unstable by establishing the existence of a negative upper bound for the lowest eigenvalue of the perturbation [12].

Note added: After this paper was submitted, we discovered yet another paper which discusses the static solutions of Einstein-Klein-Gordon [13].

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