Black Hole "No-Hair" Theorems and Numerical Relativity

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What kinds of solutions to the Einstein equations might we expect to find?

- Black hole solutions

  These have a smooth event horizon, are non-singular outside the horizon and becoming asymptotically flat far from the horizon.

- "Soliton" or "Particle"-like solutions

  These are globally regular, *i.e.* they have no singularities and become asymptotically flat at large distances.

- Others
  - Cosmological
  - Gravitational waves

The No-Hair Conjecture

- Birkhoff – Schwarzschild is the unique spherically symmetric solution with asymptotic flatness.

- Israel – Schwarzchild is the unique solution for static, nonrotating black holes with asymptotic flatness.

- Robinson - Carter – Kerr(-Newman) is the unique solution for axisymmetric, stationary black holes with asymptotic flatness.

- Price's Theorem – Everything that can be radiated away will be radiated away in collapse.

$\Rightarrow$ Black holes are completely characterized by M, J, Q_e, and Q_m (their gauge charges).

No Solitons

- Lichnerowicz – There are no globally regular solutions to the Einstein-Maxwell equations. (nonsingular and asymptotically flat)

- This was generalized to Kaluza-Klein and supergravity models.

- Deser - Coleman – There are no static Yang-Mills solutions in flat spacetime.

- Deser – There are no static soliton solutions of the Einstein-Yang-Mills equations in 2+1.

$\Rightarrow$ This is suggestive that there are no solitons in Einstein's theory.

Nonetheless ...

- Bartnik and McKinnon (1988) – There are particle-like (soliton) solutions to the EYM equations.

- Assume static, spherically symmetric, and an SU(2).

- The metric is

$$ds^2 = -A(r)^2\mu(r)dt^2 + \frac{1}{\mu(r)}dr^2 + r^2 d\Omega^2$$

- The SU(2) connection (matrix valued) is

$$\begin{aligned} A &= a(r)\tau_3 dt + w(r)\tau_1 d\theta \\ &\quad +(\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\phi \end{aligned}$$

- Finite energy $\Rightarrow a \equiv 0$.

- BC's for our ODE's: regularity at the origin

$$w(r) \sim 1 - br^2 + O(r^4)$$

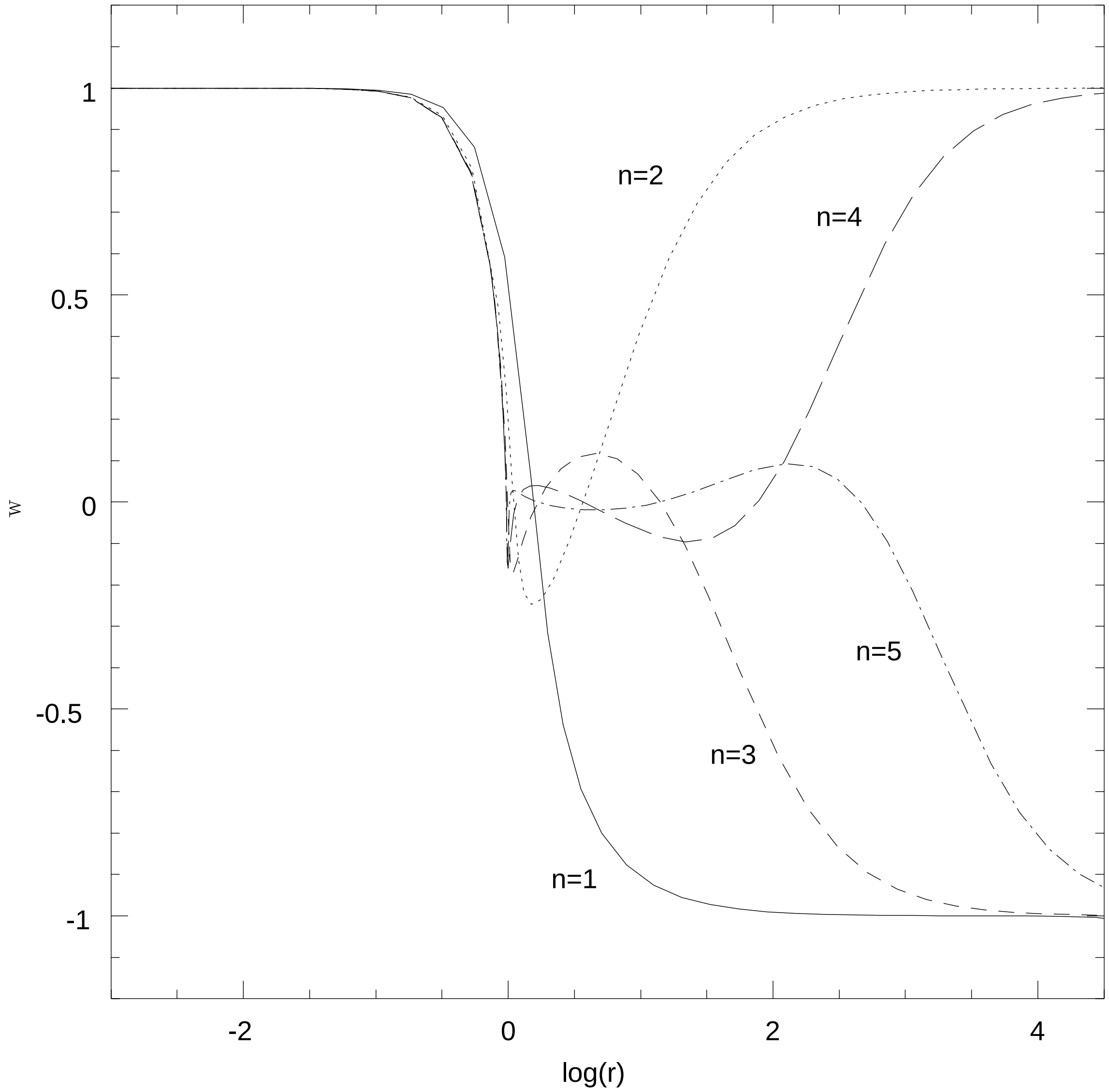
  and asymptotic flatness

$$w(r)^2 \to 1$$

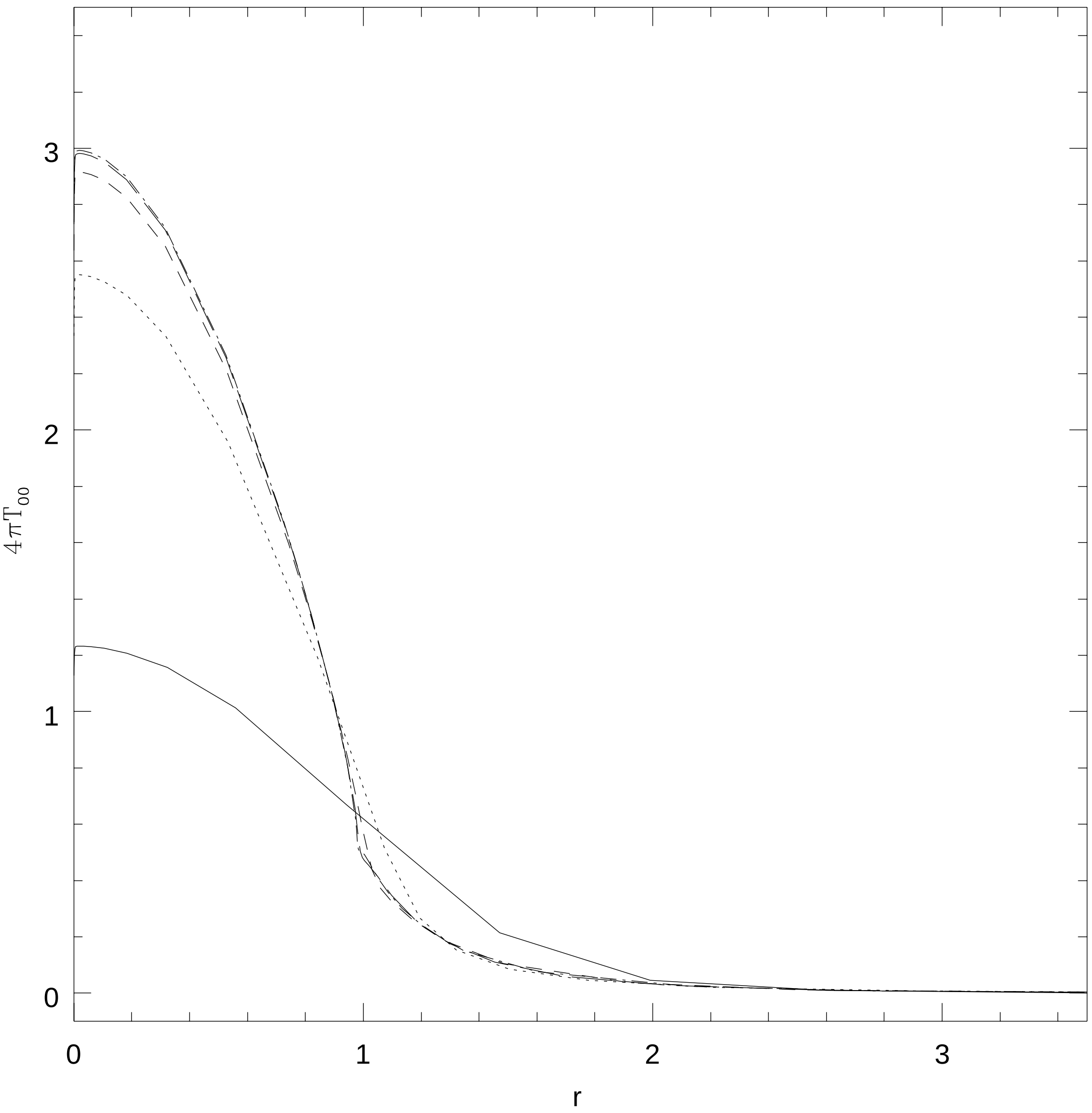
- The numerical problem is a simple shooting on b.

- Infinite number of solutions characterized by the number of zeros of $w(r)$: b_n.

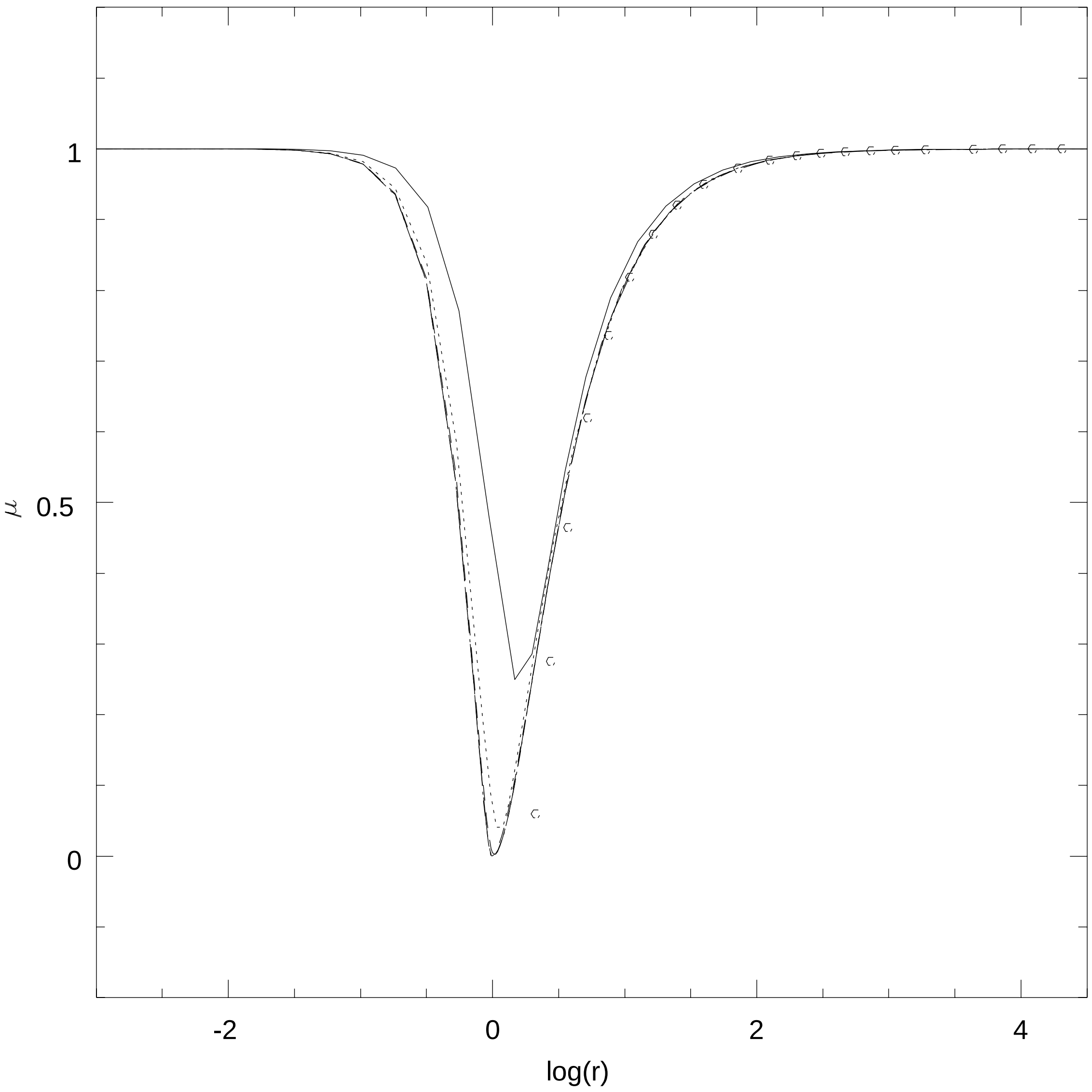
The first five BM solutions.



Energy density of the first five BM solutions.



Metric function $\mu(r)$ for the first five BM solutions



Black holes in EYM

- There also turn out to be black holes in addition to the BM "solitons."

- The requirements are almost the same
  - static, spherically symmetric metric
  - spherically symmetric SU(2) gauge connection
  - asymptotic flatness

- Added assumption is the existence of a horizon at $r_h > 0$. The place where $\mu(r_h) = 0$.

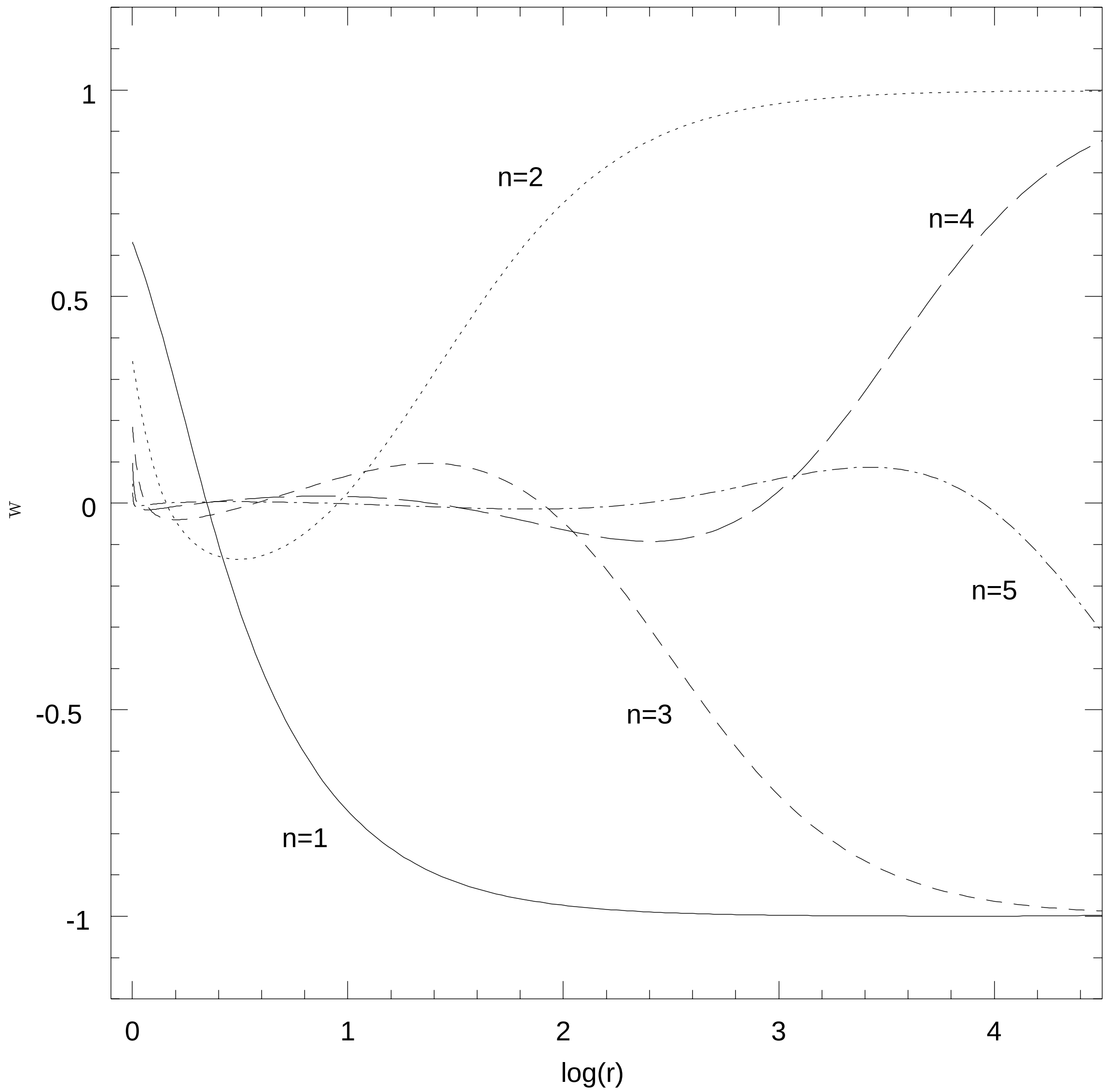
- Again, simple shooting on $w_h \equiv w(r_h)$ reveals an infinite number of discrete solutions characterized by the number (n) of zeros of $w(r)$.

- These solutions are parameterized by r_h, or correspondingly, the mass of the black hole: $M(r_h)$.

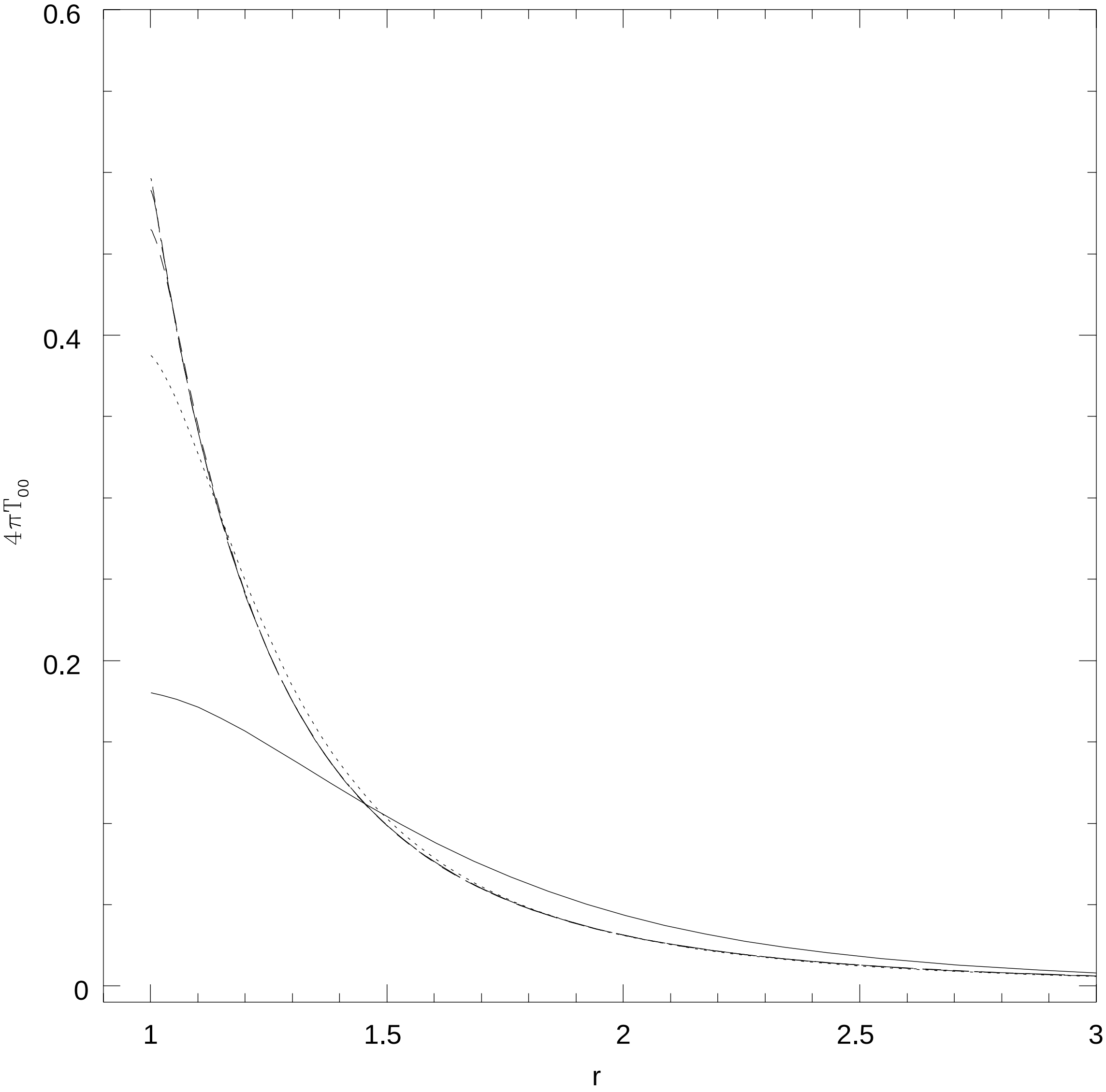
- As with the BM solutions, the global YM charge vanishes.

- In the $n \to \infty$ limit, this sequence of black hole solutions approaches the Reissner-Nordstrom solution.

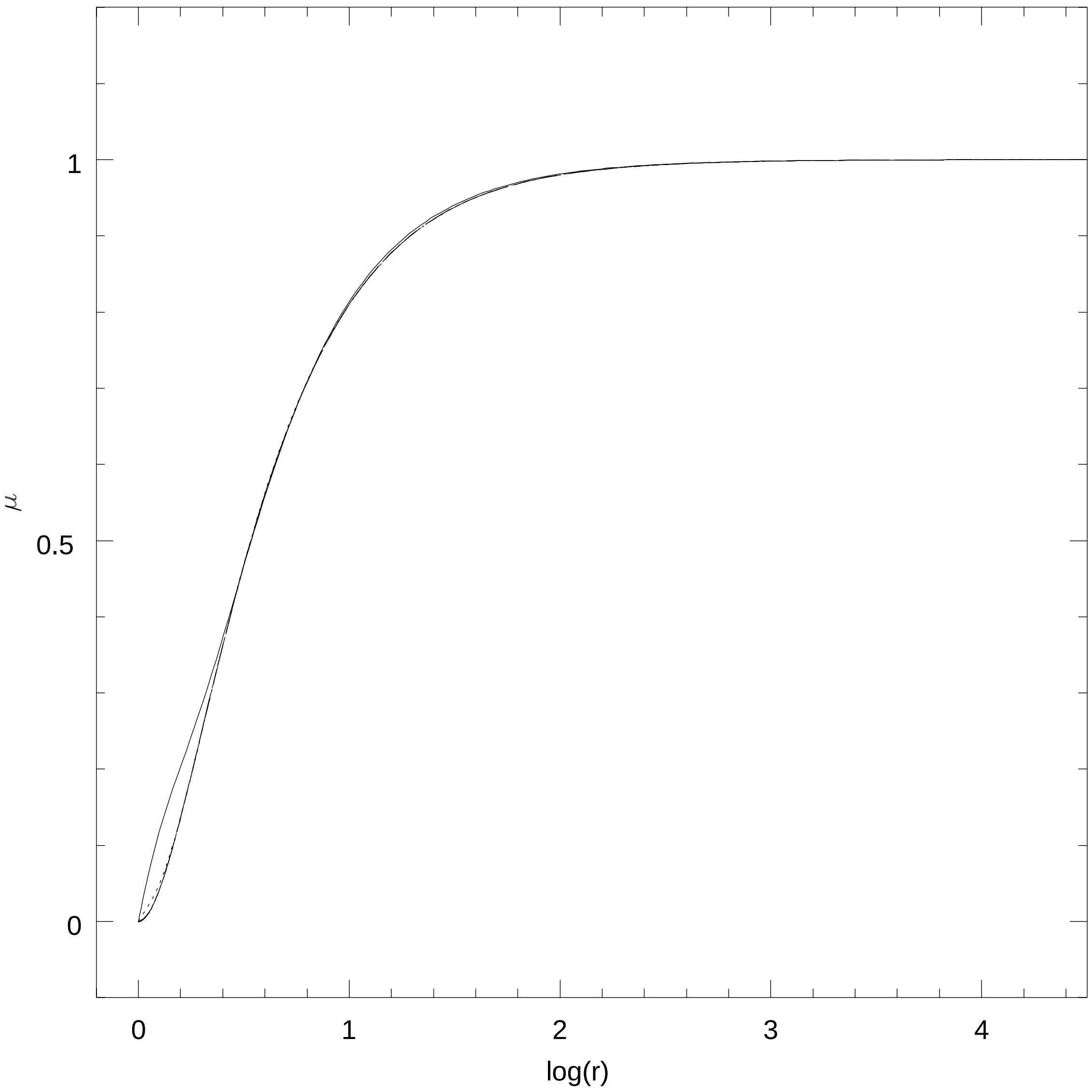
The first five EYM black hole solutions.



Energy density of the first five EYM black hole solutions.



Metric function $\mu(r)$ for the first five EYM black holes



Stability

- These solutions – both soliton and black hole – are unstable in linear perturbation theory.

- For the n^{th} solution, there are $2n$ unstable modes.

- There are n modes unstable to gravitational perturbations and n modes unstable to gauge field perturbations.

- Zhou and Straumann: BM solutions unstable either to dispersal of the fields to infinity or collapse to a Schwarzchild black hole.

- Choptuik *et al* showed that for $n = 1$ this is Type I critical behavior and that the model also has Type II critical behavior.

Einstein-Yang-Mills-some-kind-of-scalar

It is natural to generalize the EYM results to a broader class of theories which *e.g.* add some scalar field coupling.

Let's consider two.

- Einstein-Yang-Mills-Dilaton

$$\mathcal{L} = R - 2\nabla_\mu\phi\nabla^\mu\phi - e^{2\gamma\phi}F^a_{\mu\nu}F^{a\mu\nu}$$

  where $F^a_{\mu\nu}$ is the SU(2) Yang-Mills field strength and γ is a dimensionless coupling constant.

- Einstein-Yang-Mills-Higgs

$$\mathcal{L} = \frac{1}{\alpha^2}R - F^a_{\mu\nu}F^{a\mu\nu} - 2D_\mu\phi^a D^\mu\phi^a - \frac{\beta^2}{\alpha^2}(\phi^a\phi^a - 1)^2$$

  where the Higgs field ϕ^a is in the adjoint representation of SU(2) and α and β are dimensionless coupling constants.

Einstein-Yang-Mills-Dilaton

- Various theoretical models (*e.g.* string theory, Kaluza-Klein, inflation, etc) suggest the existence of a massless, real scalar field – the “dilaton.”

- The limit $\gamma \to 0$ (with a constant ϕ) recovers the Bartnik-McKinnon solutions.

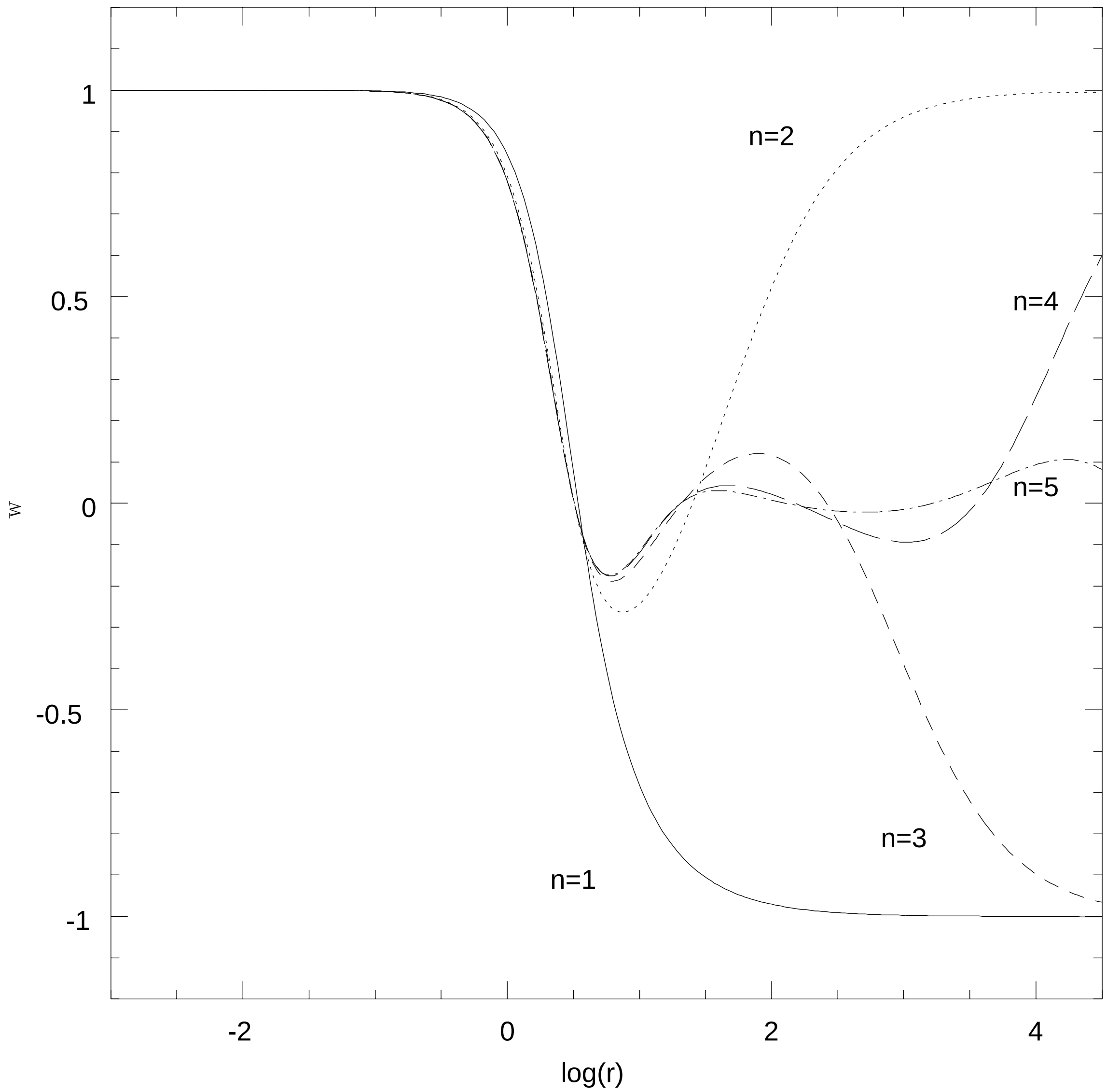
  In addition, the limit $\gamma \to \infty$ leads to YMD uncoupled from gravity which also possesses soliton solutions.

- The particular value $\gamma = 1$ describes the low-energy limit of heterotic string theory.

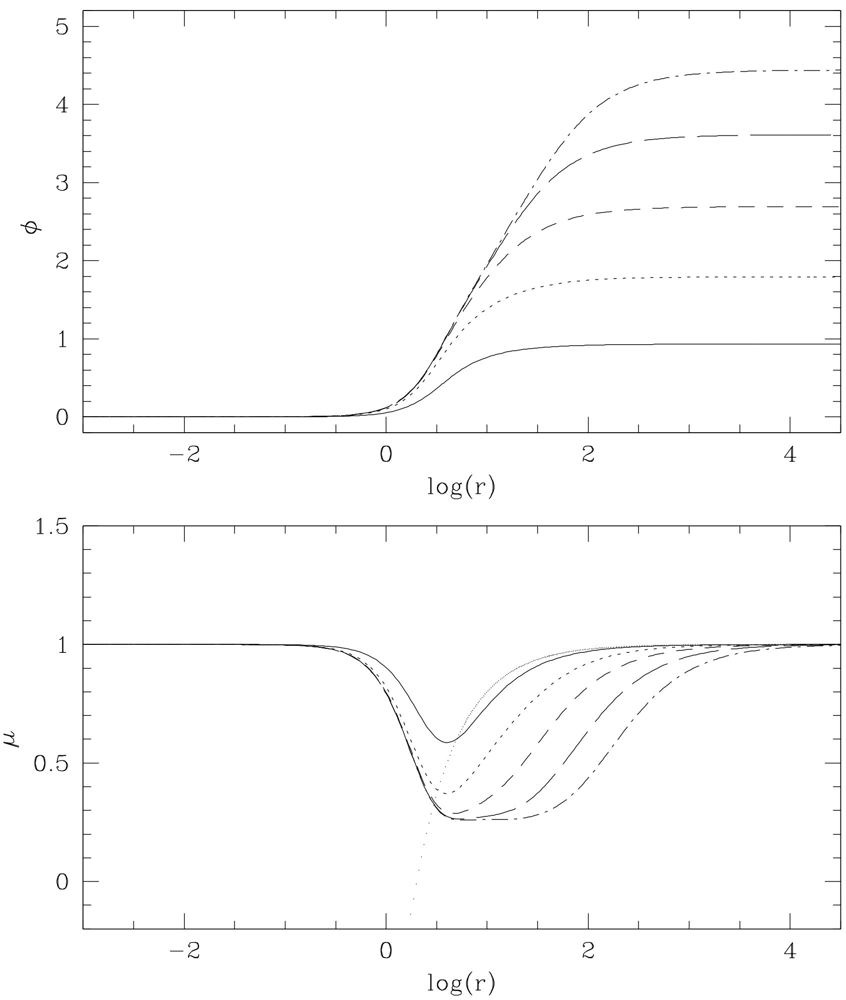
- Both solitons and black hole solutions can be found with the coupling γ describing a family of solutions.

- The assumptions and procedure are the same as before.
  - We shoot on a single parameter b, find an infinite set of discrete solutions (n)for a given γ value.

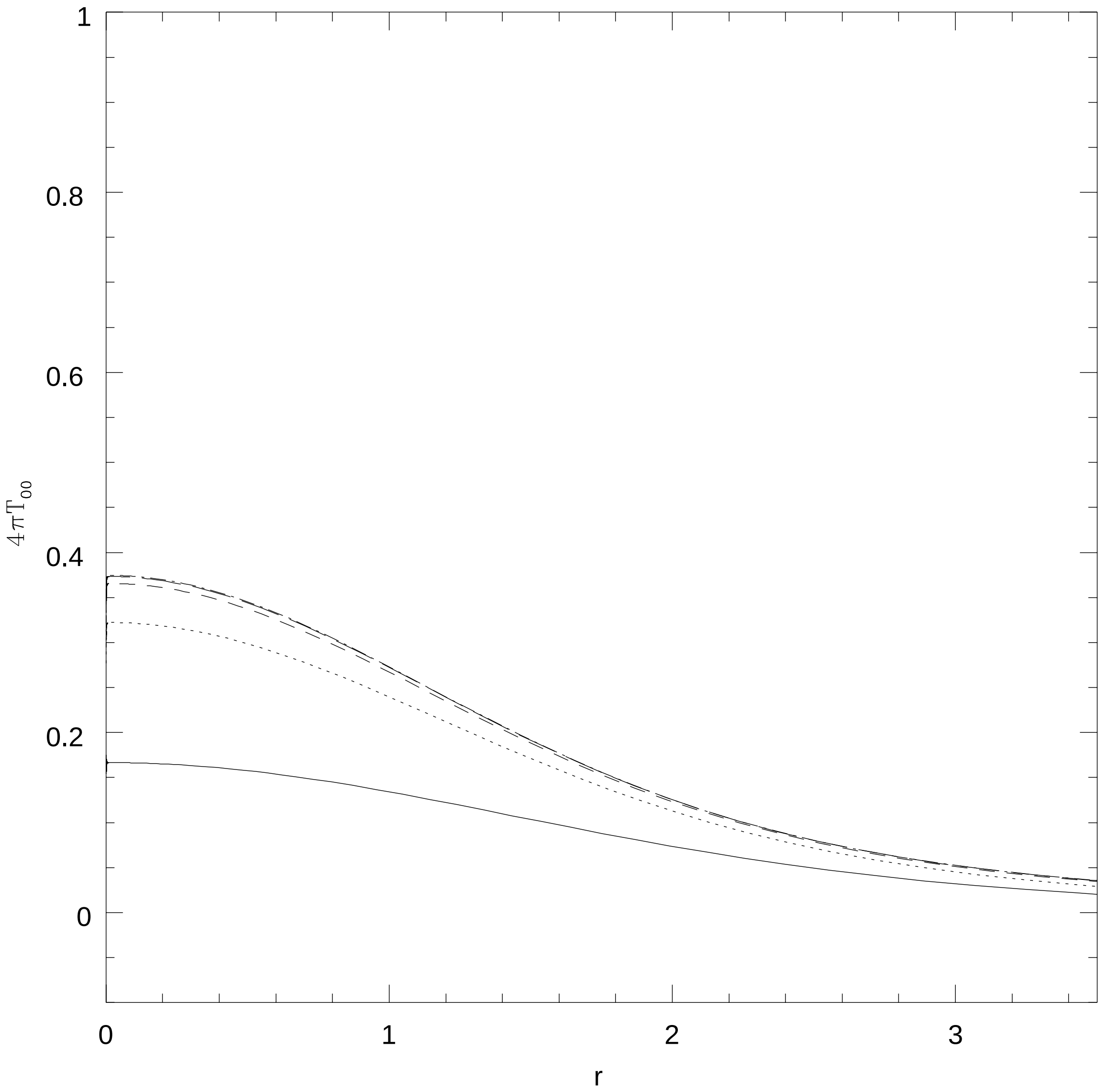
First five regular solutions of EYMD with $\gamma = 1$



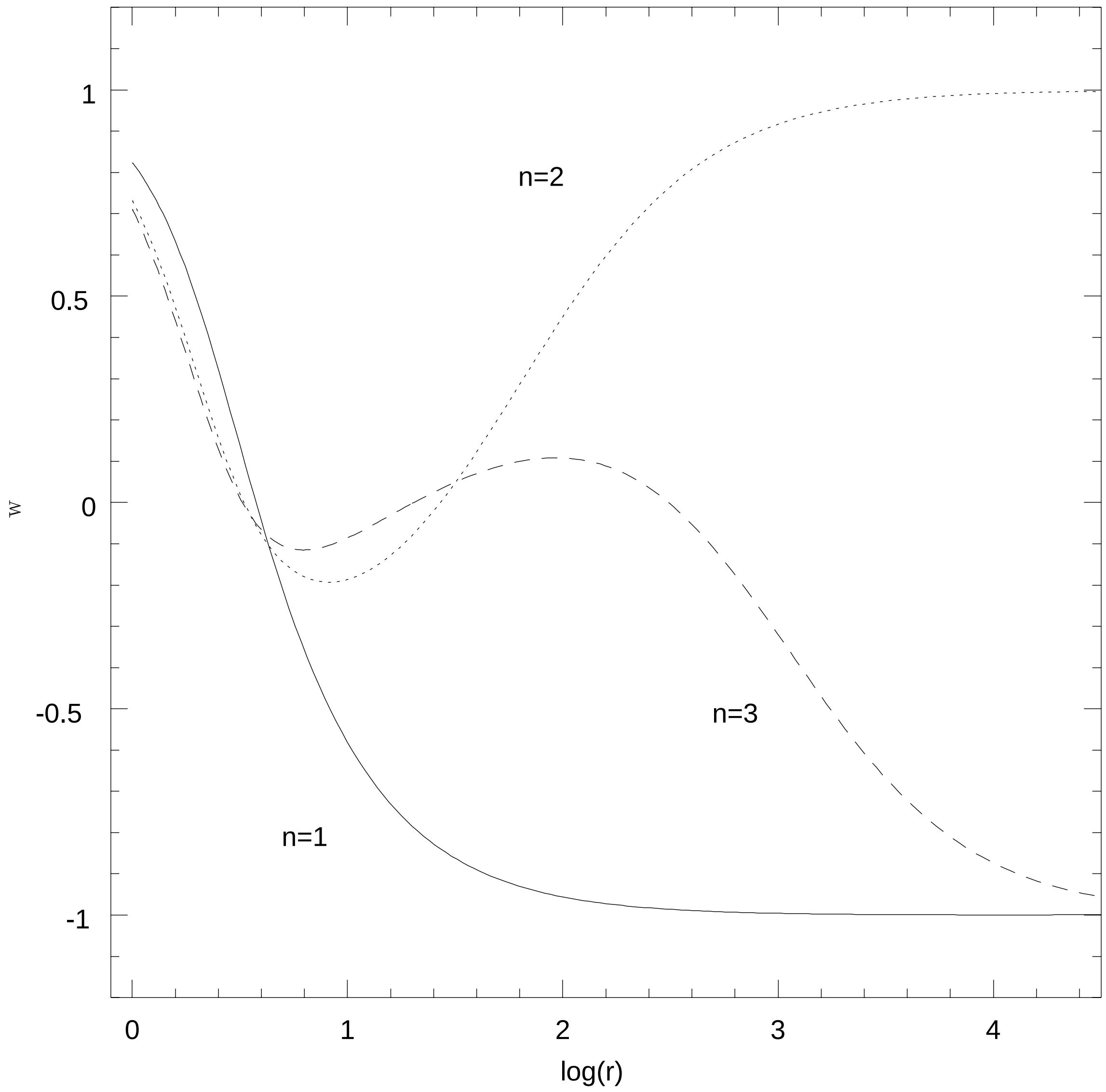
Dilaton field and metric function μ for regular solutions of EYMD with $\gamma = 1$



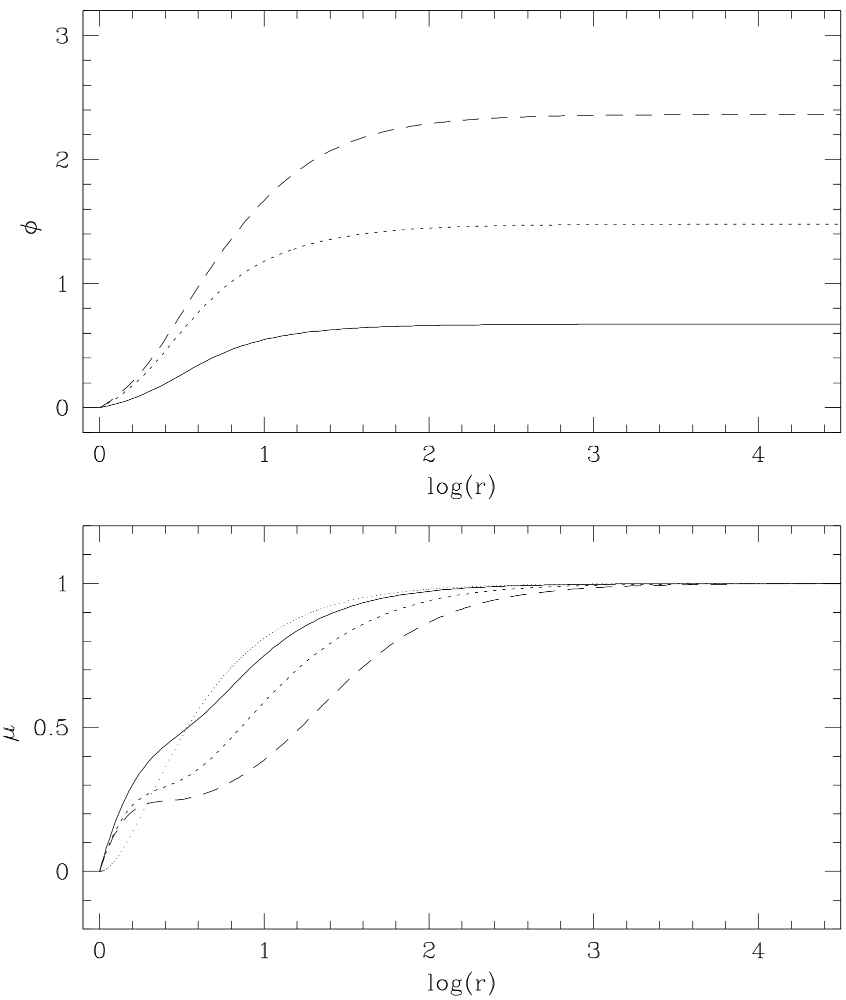
Energy density for regular solutions of EYMD with $\gamma = 1$



Black hole solutions of EYMD with $\gamma = 1$



Dilaton field and metric function μ for black hole solutions of EYMD with $\gamma = 1$



Stability of EYMD solitons and black holes

- All the solutions are also are unstable in linear perturbation theory.

- Again, for the n^{th} solution, there are $2n$ unstable modes with n modes unstable to gravitational perturbations and n modes unstable to gauge field perturbations.

- Conjecture: Like the BM soliton solutions, these will be unstable either to dispersal of the fields to infinity or collapse to a Schwarzchild black hole.

- In addition, if the BM solutions are any guide, and they belong in this family of solutions, this model should exhibit both Type I and Type II critical behavior. So we would have yet another parameterized family of critical solutions.

Einstein-Yang-Mills-Higgs

- These should describe gravitating monopoles and dyons for example, so we sort of expect to find these when gravity is "turned on."

- In addition, one might expect that once these objects become sufficiently massive, they will "collapse" and form black holes.

- We again make the assumptions of spherical symmetry, staticity, SU(2) and asymptotic flatness. In addition, we make the ansatz (hedgehog) for the Higgs field of $\phi^a = \hat{r}^a H(r)$.

- Again, assuming regularity at the origin leads to solitons and assuming the existence of a horizon (*i.e.* of r_h such that $\mu(r_h) = 0$) leads to black hole solutions.

- The numerical problem is somewhat more difficult as we must search on two parameters.

Gravitating monopoles

- We again get an infinite number ($n = 0, 1, 2...$) of monopole solutions each parameterized by α and β. The limit $\alpha \to 0$ for $\beta = 0$ and $n > 0$ correspond to the BM solutions. The $n = 0$ solutions can be thought of as the gravitating generalization of the t'Hooft-Polyakov monopole.

- The parameter α is roughly the ratio of the monopole mass to the planck mass, so as it increases we expect solutions to no longer exist as they become unstable. Indeed we get RN + throat + smooth origin.

- All the excited monopole solutions exhibit this behavior as well *i.e.* they are unstable above a critical value of their mass (α_c).

  In addition they are unstable to gravitational perturbations as well. So we can conjecture again that critical behavior will be present in this system as well.

- The lowest lying ($n = 0$) monople, is however stable to gravitational perturbations for α less than its critical value.

Non-abelian black holes

- There are also magnetically charged black holes paramterized by their radius r_h (or equivalently their mass $M(r_h)$).

- Again, β can take on any value but black holes will only exist for a range of α.

- In some regions we find colored black holes. In others we get abelian RN black holes. There are also cases where we get infinitely many – a veritable zoo.

- The majority of the solutions turn out to also be unstable to gravitational perturbations. (?)

Some conclusions and possible directions

- EYM, EYMD, and EYMH yield non-trivial solutions which can be characterized as solitons or black holes. However, the majority are unstable.
- Implications for “No-Hair Conjecture”
- Physical realization? – early universe?
- Just an introduction – much more
  - Dyonic configurations – magnetic and electric charge
  - Other gauge groups (SU(n), SO(n)...)
  - Axisymmetry, rotation ...
  - SUSY
  - Black holes supporting defects
  - Critical behavior
  - Evolve the time-dependent equations and consider non-linear stability.
  - Analytic results
  - Rigorous proofs of existence