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Constrained Evolution and Differential Boundary Conditions

(The sand box for a young numerical relativist)

Proposed questions

- How do we define constrained evolution?
- How many boundary conditions are needed for constrained evolution?
- Can there be boundary conditions other than Neumann and Dirichlet?
- How energy estimates motivate boundary conditions?

What is constrained evolution?

$$\partial_t^2 u_i = \partial^j \partial_j u_i$$
$$\partial^j u_j = 0$$

$$\partial_t^2 u_i = \partial^j \partial_j u_i$$
$$\partial^j u_j = 0$$

+ full set of ID
+ full set of BD
=
overdetermined

$$\partial_t^2 u_i = \partial^j \partial_j u_i$$
$$\partial^j u_j = 0$$

+ compatible ID
+ compatible BD
=
hope for constra-int preservation

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j$$
$$\partial^j u_j = 0$$

+ compatible ID
+ ??? many BD
=
well-posed ???

The 1D example

$$\partial_t^2 u = \partial_x^2 u$$

$$\partial_x u = 0$$

How many boundary conditions? (free evolution)

(D.Arnold, N.Tarfulea, A.Alekseenko)

- Consider the constraint quantity $C = \partial_x u$.
- Notice that C satisfies the wave equation $\partial_t^2 C = \partial_x^2 C$.
- Require either $C(a) = C(b) = 0$, $\partial_x u(a) = \partial_x u(b) = 0$

  or $\partial_x C(a) = \partial_x C(b) = 0$, $\partial_x^2 u(a) = \partial_x^2 u(b) = 0$, $\partial_t^2 u(a) = \partial_t^2 u(b) = 0$, $u(a) = u(b) = 0$

The 1D example

$$\partial_t^2 u = \partial_x^2 u$$

$$\partial_x u = 0$$

How many boundary conditions? (free evolution)

(G.Calabrese, J. Pullin, O. Sarbach, M. Tiglio, O. Reula)

- Consider the constraint quantity $C = \partial_x u$.
- Notice that C satisfies the wave equation $\partial_t^2 C = \partial_x^2 C$.
- Require, for example,

$$\partial_x C(a) = 0, \qquad C(b) = 0$$

$$\partial_x^2 u(a) = 0, \qquad \partial_x u(b) = 0$$

$$\partial_t^2 u(a) = 0,$$

$$u(a) = u_0(a)t + u_1(a)$$

The 1D example

$$\partial_t^2 u = \partial_x^2 u \tag{1}$$
$$\partial_x u = 0$$

Needs no boundary conditions!

Indeed:

- Replace with

$$\partial_t^2 u = 0 \tag{2}$$
$$\partial_x u = 0$$

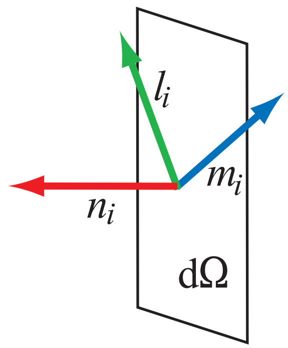
- Integrate (2) from any consistent initial data:

$$u(x,t) = u_0(x)t + u_1(x), \qquad \partial_x u_0 = \partial_x u_1 \equiv 0$$

Vector wave equation (constrained evolution)

$$\partial_t^2 u_i = \partial^j \partial_j u_i \tag{3}$$

$$\partial^j u_j = 0$$



How many boundary conditions?

Two, if can eliminate "constraint dependency."

Subtracting $\partial_i \partial^j u_j$ from (3),

$$\partial_t^2 u_i = 2\partial^j \partial_{[j} u_{i]} \tag{4}$$

$$\partial^j u_j = 0$$

Two boundary conditions:

$$n_{[i} u_{j]} = 0 \qquad \text{or} \quad u_i m^i = 0,\ u_i l^i = 0.$$

Proof of $n_{[i}u_{j]} = 0$: Contract (4) with $\partial_t u^i$, integrate,

$$\int_\Omega (\partial_t^2 u_i)\partial_t u^i = \int_\Omega 2(\partial^j \partial_{[j} u_{i]})\partial_t u^i$$

Integrating by parts:

$$\frac{1}{2}\partial_t \left[\|\partial_t u_i\|^2 + 2\|\partial_{[j} u_{i]}\|^2 \right] = \int_{\partial\Omega} (\partial_{[j} u_{i]}) n^{[j} \partial_t u^{i]}$$

The rest follows from the identity:

$$\|\partial_{(i} u_{j)}\|^2 = \|\partial_{[i} u_{j]}\|^2 + \int_{\partial\Omega} (\partial_j n^i u_i) u^j + \|\partial^i u_i\|^2 - \int_{\partial\Omega} u_i n^i \partial^j u_j$$

(check $u_i m^i = u_i l^i = 0$, thus $(\partial_j n^i u_i) u^j = (\frac{\partial}{\partial n} u_i n^i) u_j n^j = 0$,
since $\partial^j u_j = \frac{\partial}{\partial n} u_i n^i + \frac{\partial}{\partial m} u_i m^i + \frac{\partial}{\partial l} u_i l^i = 0$.)

Inhomogeneous Dirichlet data $n_{[i}u_{j]} = g_3 n_{[i}m_{j]} + g_2 n_{[i}m_{j]}$
is equivalent to

$$u_j m^j = g_2, \quad u_j l^j = g_3, \quad \text{implies } \frac{\partial}{\partial n} u_j n^j = -\frac{\partial}{\partial m} g_2 - \frac{\partial}{\partial l} g_3$$

(from $\partial^i u_i \equiv 0$).

Radiation type condition $(\partial_{[i}u_{j]})n^{[i}\partial_t u^{j]} \leq 0$, for example,

$$\frac{\partial}{\partial n} u_j m^j + \partial_t u_j m^j = \frac{\partial}{\partial m} u_j n^j, \quad \frac{\partial}{\partial n} u_j l^j + \partial_t u_j l^j = \frac{\partial}{\partial l} u_j n^j.$$

$$\text{(implies)} \quad \partial_t(\partial_t u_j n^j + \frac{\partial}{\partial n} u_j n^j) = 0$$

(from $\partial^i u_i \equiv 0$, $\frac{\partial}{\partial n}\partial^j u_j = 0$, commuting derivatives and (3))

Differential BCs (conserving $\partial^i u_i = 0$):

$$\boldsymbol{\partial_t^2 u_i \;=\; \partial^j \partial_j u_i}$$

give constraint compatible $\boldsymbol{u_i(0)}$, $\boldsymbol{\partial_t u_i(0)}$ and

$$\boldsymbol{n_{[i} u_{j]} = 0} \quad \text{and} \quad \boldsymbol{\partial^j u_j = 0}$$

Proof. Notice that $C = \partial^j u_j$ satisfies the wave equation $\partial_t^2 C = \partial^j \partial_j C$, with $C(0) = \partial_t C(0) = 0$.

The second boundary condition implies $C|_{\partial\Omega} = 0$. Thus,

$$\boldsymbol{C \equiv 0, \quad \Rightarrow \quad \partial^j u_j \equiv 0}$$

Verify $n_{[i} u_{j]} = 0$ as in the previous example.

More fancy BCs:

$$\partial_t^2 u_i \;=\; \partial^j \partial_j u_i$$

give constraint compatible $u_i(0)$, $\partial_t u_i(0)$ **and**

$$\partial_{[i} u_{j]} n^i = 0 \quad \text{and} \quad \partial^j u_j = 0$$

(Motivated by the energy identity

$$\frac{1}{2}\partial_t\left[\|\partial_t u_i\|^2 + 2\|\partial_{[j} u_{i]}\|^2\right] = \int_{\partial\Omega} (\partial_{[j} u_{i]}) n^{[j} \partial_t u^{i]})$$

Reduces to ($\rho = \partial_t u_i(0) n^i t + u_i(0) n^i$)

$$u_i n^i = \rho, \quad \frac{\partial}{\partial n} u_i m^i = \frac{\partial}{\partial m}\rho, \quad \frac{\partial}{\partial n} u_i l^i = \frac{\partial}{\partial l}\rho.$$

Appendix

Linearized BSSN equations

$$\partial_t \varphi = -\tfrac{1}{6}\kappa + \tfrac{1}{6}\partial^s \beta_s, \qquad \partial_t \alpha = -\kappa, \qquad \partial_t \kappa = -\partial^l \partial_l \alpha,$$

$$\partial_t \tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + 2\partial_{(i}\beta_{j)} - \tfrac{2}{3}\delta_{ij}\partial^s \beta_s,$$

$$\partial_t A_{ij} = \tfrac{1}{2}\partial^l \partial_l \tilde{\gamma}_{ij} + \partial_{(i}\Gamma_{j)} - 2\partial_i \partial_j \varphi - 2\delta_{ij}\partial^l \partial_l \varphi - \partial_i \partial_j \alpha + \tfrac{1}{3}\delta_{ij}\partial^l \partial_l \alpha,$$

$$\partial_t \Gamma_i = -\tfrac{4}{3}\partial_i \kappa + \tfrac{1}{3}\partial_i \partial^s \beta_s + \partial^l \partial_l \beta_i,$$

Constraint equations:

$$\partial^p \partial^q \tilde{\gamma}_{pq} - 8\partial^l \partial_l \varphi = 0, \quad \text{and/or} \quad \partial^l \Gamma_l - 8\partial^l \partial_l \varphi = 0$$

$$\partial^j A_{ij} - \tfrac{2}{3}\partial_i k = 0,$$

$$\Gamma_j = \partial^l \tilde{\gamma}_{lj}.$$

Reduction to second order in time

$$\partial_t^2 A_{ij} = \partial^l \partial_l A_{ij}$$
$$\partial_t^2 k = \partial^l \partial_l k.$$

Introduce

$$M_i = \partial^j A_{ij} - \frac{2}{3}\partial_i k$$

Evolution of constraint M_i:

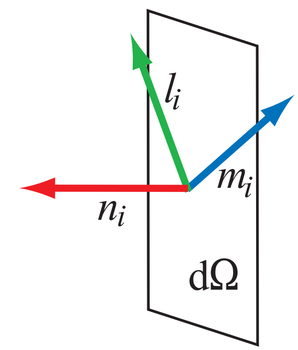
$$\partial_t^2 M_j = \partial^l \partial_l M_j$$

Initial data $M_j(0) = \partial_t M_i(0) = 0$ for compatible data. Thus, $M_i \equiv 0$ as long as

$$(\frac{\partial}{\partial n} M_j)\partial_t M^j \leq 0 \quad \text{on} \quad \partial\Omega$$

$(\frac{\partial}{\partial n}M_j)\partial_t M^j = 0$ via the main variables.

Introduce, orthonormal basis: n_i, m_i, l_i,



Rewrite, $\partial^j u_j = \frac{\partial}{\partial n} u_i n^i + \frac{\partial}{\partial m} u_i m^i + \frac{\partial}{\partial l} u_i l^i$,

$$\partial_i \kappa = \frac{\partial}{\partial n}\kappa n_i + \frac{\partial}{\partial m}\kappa m_i + \frac{\partial}{\partial l}\kappa l_i$$

Decompose

$$A_{ij} = A1\,(n_{(i}m_{j)}) + A2\,(n_{(i}l_{j)}) + A3\,(l_{(i}m_{j)})$$

$$+ A4\,(l_i l_j - m_i m_j) + A5\,(2n_i n_j - l_i l_j - m_i m_j).$$

Substitute into $M_i = \partial^j A_{ij} - \frac{2}{3}\partial_i k$ and the result into $(\frac{\partial}{\partial n}M_j)\partial_t M^j = 0$

Simplify ...

$$
\begin{aligned}
&(\frac{\partial}{\partial n}M_i)(\partial_t M^i) \\
= \;& \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa] \\
& \times \partial_t[\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa] \\
+ \;& \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa] \\
& \times \partial_t[\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa] \\
+ \;& \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa] \\
& \times \partial_t[\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa](= 0\ ??
\end{aligned}
$$

By a direct observation, either of the two sets of boundary conditions are constraint-preserving:

$$A1 = 0, \quad A2 = 0, \quad \frac{\partial}{\partial n} A3 = 0, \quad \frac{\partial}{\partial n} A4 = 0,$$
$$\frac{\partial}{\partial n} A5 = 0, \quad \frac{\partial}{\partial n} \kappa = 0,$$

$$\frac{\partial}{\partial n} A1 = 0, \quad \frac{\partial}{\partial n} A2 = 0, \quad A3 = 0, \quad A4 = 0,$$
$$A5 = 0, \quad \kappa = 0.$$

(the first set eliminates the second multiplier in the first term of $((\partial/\partial n)M^i)(\partial_t M_i) = 0$ and the first multipliers in the second and third terms (by commuting partial derivatives and using evolution eqn. The second set is verified in a similar way.)

Too restrictive!

Differential boundary conditions

Require $M_i = 0$ on $\partial\Omega$:

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa &= M_i n^i = 0 \\
\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa &= M_i m^i = 0 \\
\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa &= M_i l^i = 0
\end{aligned}$$

For example, prescribe $A3$, $A4$, κ, use $M_i = 0$ as the boundary conditions for the rest.

Well-posedness ???

Evolving boundary conditions

Require $\frac{\partial}{\partial n} M_i n^i = 0, \quad M_i l^i = 0, \quad M_i m^i = 0$. Solve for

$$2\frac{\partial^2}{\partial t^2}A5 - (\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial m^2})A5 - \frac{2}{3}\frac{\partial^2}{\partial t^2}\kappa + \frac{4}{3}(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial m^2})\kappa$$
$$= (\frac{\partial^2}{\partial l^2} - \frac{\partial^2}{\partial m^2})A4 + \frac{\partial}{\partial l}\frac{\partial}{\partial m}A3,$$

$$\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa = 0$$
$$\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa = 0$$

For example, prescribe $A3$, $A4$, κ, evolve first equation for $A5$, use other two as the inhomogeneous Neumann data on $A1$ and $A2$.

The static constraints equations:

$$\partial_t^2 A_{ij} = \partial^l \partial_l A_{ij} - 2\partial_{(i} M_{j)}$$

$$\partial_t^2 \kappa = \partial^l \partial_l \kappa - \frac{3}{2}\partial^l M_l$$

Implies

$$\partial_t^2 M_i \equiv 0!!!$$

NO NEED IN
CONSTRAINT-PRESERVING BOUNDARY
CONDITIONS!

Linearized BSSN, densitized lapse ($\alpha = 6\varphi$, $\beta_i = 0$)

$$\begin{aligned}
\partial_t \varphi &= -\frac{1}{6}\kappa, \\
\partial_t \kappa &= -6\partial^l \partial_l \varphi, \\
\partial_t \tilde{\gamma}_{ij} &= -2A_{ij}, \\
\partial_t A_{ij} &= -\frac{1}{2}\partial^l \partial_l \tilde{\gamma}_{ij} + \partial_{(i}\Gamma_{j)} - 8\partial_i \partial_j \varphi, \\
\partial_t \Gamma_i &= -\frac{4}{3}\partial_i \kappa.
\end{aligned}$$

Constraint equations:

$\partial^p \partial^q \tilde{\gamma}_{pq} - 8\partial^l \partial_l \varphi = 0,$ and/or $\partial^l \Gamma_l - 8\partial^l \partial_l \varphi = 0;$

$\partial^j A_{ij} - \frac{2}{3}\partial_i k = 0;$

$\Gamma_j = \partial^l \tilde{\gamma}_{lj}.$

Energy estimate with boundaries

(C.Gundlach and J.M.Martin-Garcia)

Growth of energy

$$\epsilon = \|\kappa\|^2 + \|A\|^2 + 36\|\partial_l\varphi\|^2 + \|\Gamma_l - 8\partial_l\varphi\|^2$$

$$+\|\frac{1}{2}\partial_l\tilde{\gamma}_{ji} - \delta_{l(i}(\Gamma_{j)} - 8\partial_{j)}\varphi)\|^2,$$

is determined by three boundary terms

$$\partial_t\epsilon = -6\int_{\partial\Omega}(\frac{\partial}{\partial n}\varphi)\kappa - \int_{\partial\Omega}(\frac{\partial}{\partial n}\tilde{\gamma}_{ij})A^{ij}$$

$$+\,2\int_{\partial\Omega} n_{(i}(\Gamma_{j)} - 8\partial_{j)}\varphi)A^{ij}.$$

Input is needed

- De Rham complex, de Rham complex, de Rham complex!
- Semigroup theory, proofs of existence in $H(\mathbf{div})$-like spaces.
- Analysis of long term stability for nonlinear equations.
- Nonlinear energy estimates.